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J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

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THE

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ON THE SEXTACTIC POINTS OF A QUARTIC.

By A. B. Basset, M.A., F.R.S.

1. A POINT where a conic has a contact of the fifth order with a curve is called by Cayley a *sextactic** point; and he has shown that the number of such points is $3n(4n-9)-24\delta-27\kappa$. It is also obvious that the number of sextactic points of a curve is the same as that of its reciprocal; a result which may also be obtained from Plücker's equations by showing that Cayley's formula is equivalent to $3m(4m-9)-24\tau-27\iota$. Hence in the case of anautotomic, trinodal and tricuspidal quartics the number of sextactic points is respectively equal to 84, 12, and 3; that is to say three times the number of the double tangents. These numbers suggest the probability of a connection between the sextactic points and the double tangents, and it is easy to show that in many cases such a connection exists.

A large number of quartics are capable of being projected into hemisymmetrical curves. In this case let the axis of x be the axis of symmetry, and let the origin O be one of the four points in which this axis cuts the curve. Then the conic $y^2=Ax^2+Bx$ obviously touches the quartic at the origin and intersects it at three pairs of points which are symmetrically situated with respect to the axis of x; and since the conic contains two independent constants these may be determined so that two pairs of points coincide with the origin, which is therefore a sextactic point. Hence the four points where the axis of x cuts the quartic are sextactic points.

Now the tangents at these four points are parallel to the axis of y, and therefore intersect at a point on the line at

* *Collected Mathematical Papers*, Vol. v., pp. 221 and 515; Vol. vi., p. 191 to 292.

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infinity; also the quartic, if anautotomic, has four double tangents parallel to this axis, which consequently intersect at the same point on the line at infinity. We thus obtain the following theorem:—

If an anautotomic quartic is capable of being projected into a hemisymmetrical curve four of the double tangents must intersect at a point, and the points of contact of the four ordinary tangents which can be drawn from this point must be collinear. Also the four last mentioned points are sextactic points.

When a quartic is capable of being projected into a symmetrical curve, four more sextactic points lie on the axis of y, and their positions on the unprojected curve can be determined in the same manner.

When the quartic is autotomic, the sextactic points in the foregoing cases must be determined by considering what becomes of the double tangents in consequence of the existence of double points.

All binodal quartics with two biflecnodes can be projected into Cassinians, which are symmetrical curves. Hence in the case of quartics of this species, two of the double tangents must intersect at a point P on the line joining the two biflecnodes, whilst two more must intersect at a point Q on the same line; and the points of contact of the four ordinary tangents which can be drawn from P and Q respectively are eight of the sextactic points.

Four more can be determined as follows. Let AB, AC be the tangents at the biflecnode A of a uninodal quartic, and let BC be the harmonic polar of A. Then the equation of the quartic is

$$\alpha^2\beta\gamma + u_4 = 0.$$

The equation $u_4 = 0$ is that of the four tangents which can be drawn from A; and if A be projected to infinity in such a manner that AB and AC are perpendicular to BC, the quartic assumes the form

$$y^2 v_2 + v_4 = 0,$$

where v_n is a polynomial in x alone. The projected curve is accordingly hemisymmetrical, and consequently the points of contact of the four tangents from the biflecnode are sextactic points. If the quartic has another double point, the harmonic polar must pass through it, and two of the tangents from A coalesce into the line joining the two nodes. Hence in the case of a quartic with two biflecnodes, four more sextactic

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points are the points of contact of the tangents drawn from the two biflecnodes*.

All bicuspidal quartics can be projected into Cartesians, which are hemisymmetrical curves; and four of the sextactic points are the points of contact of the four ordinary tangents which can be drawn from the point where the double tangent intersects the line joining the cusps.

I shall now completely determine the sextatic points when (1) the quartic has three biflecnodes, (2) when it is nodo-bicuspidal, (3) when it is tricuspidal.

2. In the first case, project the quartic into a lemniscate, and we obtain the theorem:—*The six points of intersection of the four double tangents lie in pairs upon the sides of the triangle formed by joining the biflecnodes. From each of these six points two ordinary tangents can be drawn to the quartic, and their twelve points of contact are the twelve sextactic points.*

Let ABC be the triangle of reference; AD, AD' the internal and external bisectors of A; also let the equation of the quartic be

$$\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The four double tangents are

$$\alpha + \beta + \gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\alpha - \beta - \gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\alpha + \beta - \gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$\alpha - \beta + \gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Equations (2) and (3) show that two of the double tangents intersect at D', and (4) and (5) show that two more intersect at D. The remaining four intersections can be determined in a similar manner. The polar cubic of D is

$$(\alpha^2 + \beta\gamma)(\beta + \gamma) = 0.$$

---

* In connection with flecnodes and biflecnodes, the following two singularities may be mentioned. The equation of a uninodal quartic having a flecnode at A is $\alpha^2 u_1 v_1 + \alpha u_1 u_2 + u_4 = 0$, and the curve is of the tenth class. Now it is possible for the two tangents at the flecnode to coincide, in which case $v_1 = u_1$, and the equation becomes $\alpha^2 u_1^2 + \alpha u_1 u_2 + u_4 = 0$. By forming the polar cubic of B, which may be any arbitrary point, it can be shown that the cubic intersects the quartic at four coincident points at A and at eight ordinary points. Hence the last curve is of the eighth class and is therefore a binodal quartic having a special kind of tacnode, such that the tacnodal point is a point of inflexion on one branch.

In the same way it can be shown that if the tangents at a biflecnode coincide, the curve is also of the eighth class, having another special kind of tacnode in which the tacnodal point is a point of inflexion on both branches. These singularities might be called *tacflecnodes* and *tacbiflecnodes* respectively.

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The conic $\alpha^2 + \beta\gamma = 0$ passes through the four points of contact of the double tangents which intersect at D, whilst the line $\beta + \gamma = 0$ intersects the quartic at P, Q, which are the points of contact of the two ordinary tangents which can be drawn from D. The coordinates of P, Q are

$$\left.\begin{aligned} \iota\alpha\sqrt{2} &= \beta = -\gamma \\ -\iota\alpha\sqrt{2} &= \beta = -\gamma \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

In the same way it can be shown that the points of contact P, Q of the two tangents which can be drawn from D' lie on AD. The remaining eight points can be determined in a similar manner.

From these results it can be shown that:

(i) *A conic can be described which passes through the six sextactic points which lie on the three external bisectors of the triangle ABC.*

(ii) *Three other conics can be described which respectively pass through the six sextactic points which lie on any two internal bisectors and the external bisector of the remaining angle.*

It is not, however, possible to describe a conic which passes through the six sextactic points which lie on the three internal bisectors; hence a cubic cannot be drawn through more than eleven arbitrarily chosen sextactic points.

3. Every quartic with three biflecnodes can be projected into the curve $a^2(x^2 + y^2) = x^2y^2$, which is the reciprocal polar of a four-cusped hypocycloid. Hence this curve also possesses twelve sextactic points, which are the points where the six lines joining the four *imaginary* nodes intersect the curve. This construction holds good in the case of all sextic curves which can be projected into a four-cusped hypocycloid.

The reciprocal polar of a two-cusped epicycloid is a quartic curve having one biflecnode and two ordinary nodes*; also since the epicycloid has two triple points of the third† kind at the circular points, the reciprocal curve has two points of undulation, and can therefore be projected into the inverse with respect to its centre of an ellipse whose major axis is

* *Cubic and Quartic Curves*, p. 210, equation (15).

† In my book on *Cubic and Quartic Curves*, I have enumerated *four* different kinds of triple points; but as the analytical treatment of curves having a triple point composed of three ordinary nodes is independent of their character, I think it better on further consideration to divide triple points into three species according as they are composed (i) of three nodes; (ii) two nodes and a cusp; (iii) two cusps and a node.

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$\sqrt{2}$ times its minor axis. Four of the sextactic points can be determined by the previous methods and the remaining eight by the parametric method.

4. By projecting a tricuspidal quartic into a three-cusped hypocycloid, it follows that the three sextactic points are the points where the cuspidal tangents cut the curve; and the same result follows by reciprocating a nodal cubic.

By projecting a nodobicuspidal quartic into a limaçon, it follows that two of the sextactic points are the vertices of the limaçon; that is to say they are the points of contact of the two ordinary tangents which can be drawn from the point where the double tangent intersects the line joining the cusps. The remaining four points may be determined as follows. Let the equation of the curve be

$$(\beta\gamma + \gamma\alpha + \alpha\beta)^2 = m^2\alpha^2\beta\gamma \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

then the coordinates of any point on the curve may be expressed in terms of a parameter θ by means of the equations

$$\alpha = -\theta^2,\ \beta = \theta^2 - m\theta + 1,\ \gamma = \theta^2(\theta^2 - m\theta + 1) \ldots\ldots (8).$$

The parameters of the node A are given by $\theta^2 - m\theta + 1 = 0$, whilst $\theta = 0$ and $\theta = \infty$ are the parameters of the cusps B and C. The sextactic points corresponding to the vertices of the limaçon lie on the line $\beta - \gamma = 0$, and their parameters are $\theta = \pm 1$.

Let the equation of any sextactic conic be

$$\lambda\alpha^2 + \mu\beta^2 + \nu\gamma^2 + \lambda'\beta\gamma + \mu'\gamma\alpha + \nu'\alpha\beta = 0 \ldots\ldots (9).$$

Substituting the values of α, β, γ from (8) in (9) we get

$$\nu\theta^8 - 2m\nu\theta^7 + \{(m^2+2)\nu + \lambda' - \mu'\}\theta^6 - (2\nu + 2\lambda' - \mu')\, m\theta^5$$
$$+ \{\nu + (m^2+2)\lambda' + \lambda + \mu - \mu' - \nu'\}\theta^4 - (2\lambda' + 2\mu - \nu')\, m\theta^3$$
$$+ \{\lambda' + (m^2+2)\mu - \nu'\}\theta^2 - 2m\mu\theta + \mu = 0 \ldots\ldots (10).$$

Let ϕ be the parameter of any sextactic point; χ_1, χ_2 the parameters of the two remaining points where the sextactic conic cuts the quartic. Then

$$6\phi + \chi_1 + \chi_2 = 2m \ldots\ldots\ldots\ldots\ldots (11),$$

$$\frac{6}{\phi} + \frac{1}{\chi_1} + \frac{1}{\chi_2} = 2m \ldots\ldots\ldots\ldots\ldots (12),$$

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$$P = 15\phi^2 + 6\phi(\chi_1 + \chi_2) + \chi_1\chi_2 = m^2 + 2 + (\lambda' - \mu')/\nu \ldots (13),$$

$$Q = \frac{15}{\phi^2} + \frac{6}{\phi}\left(\frac{1}{\chi_1} + \frac{1}{\chi_2}\right) + \frac{1}{\chi_1\chi_2} = m^2 + 2 + (\lambda' - \nu')/\mu \ldots\ldots (14),$$

$$R = 20\phi^3 + 15\phi^2(\chi_1 + \chi_2) + 6\phi\chi_1\chi_2 = 2m + (2\lambda' - \mu')\, m/\nu \ldots (15),$$

$$S = \frac{20}{\phi^3} + \frac{15}{\phi^2}\left(\frac{1}{\chi_1} + \frac{1}{\chi_2}\right) + \frac{6}{\phi\chi_1\chi_2} = 2m + (2\lambda' - \nu')\, m/\mu \ldots (16),$$

$$\phi^6\chi_1\chi_2 = \mu/\nu \ldots\ldots\ldots\ldots\ldots (17),$$

whence

$$R - mP + m^3 = \lambda' m/\nu,$$

$$S - mQ + m^3 = \lambda' m/\mu,$$

accordingly by (17)

$$R - mP + m^3 = (S - mQ + m^3)\,\phi^6\chi_1\chi_2 \ldots\ldots (18).$$

By means of (11) and (12) the values of P, Q, R, S may be written

$$P = 12m\phi - 21\phi^2 + \chi_1\chi_2,$$

$$Q = \frac{12m}{\phi} - \frac{21}{\phi^2} + \frac{1}{\chi_1\chi_2},$$

$$R = 30m\phi^2 - 70\phi^3 + 6\phi\chi_1\chi_2,$$

$$S = \frac{30m}{\phi^2} - \frac{70}{\phi^3} + \frac{6}{\phi\chi_1\chi_2},$$

also by (11) and (12)

$$\chi_1\chi_2 = \frac{m - 3\phi}{m - 3/\phi}.$$

Substituting these values in (18) we obtain

$$\{m\phi^6 - 6\phi^5 - 70\phi^3 + 51m\phi^2 - 12m^2\phi + m^3\}(m\phi - 3)$$
$$= \{m - 6\phi - 70\phi^3 + 51m\phi^4 - 12m^2\phi^5 + m^3\phi^6\}\,\phi\,(m - 3\phi) \ldots (19).$$

The form of (19) shows that if $\phi = \alpha$ is a root, $\phi = \alpha^{-1}$ is another root; also that it is divisible by $\phi^2 - 1$, which gives the parameters of the two sextactic points already determined. Dividing out by $\phi^2 - 1$, and writing

$$y = m(\phi + \phi^{-1}) \ldots\ldots\ldots\ldots\ldots (20),$$

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(19) after some reduction becomes

$$3y^3-(m^2+35)\,y^2+6\,(m^2+24)\,y+m^4-16m^2-192=0 \ldots\ldots(21).$$

This is a sextic equation for determining ϕ, whereas the required equation ought to be a quartic equation for ϕ. Now each stationary tangent taken twice satisfies the analytical condition of a sextactic conic, and the parameters of the two points of inflexion are given by the equation

$$m^2-3m\,(\phi+\phi^{-1})+8=0,$$

or

$$3y-m^2-8=0,$$

and it will be found that the left hand side of (21) is divisible by this factor, leaving

$$y^2-9y+24-m^2=0 \ldots\ldots\ldots\ldots\ldots\ldots(22),$$

which determines the remaining four sextactic points.

When $m=2$, the quartic becomes tricuspidal and $\phi=1$ is a double root of (22), whilst the remaining two roots are $\phi=\frac{1}{2}$, $\phi=2$. In this case three of the sextactic points coincide with the double point A which becomes a cusp, whilst the other three roots -1, $\frac{1}{2}$, and 2 are the parameters of the points where the cuspidal tangents intersect the curve. Accordingly when m does not differ much from 2, there are two sextcatic points in the neighbourhood of the node A, and two more in the neighbourhood of the points where the cuspidal tangents at B and C respectively intersect the curve. The positions of the remaining two points $\phi=\pm1$ have already been determined.

In the general case of any quartic with three double points it would be possible by means of the parametric method to obtain the duodecimic equation which determines the parameters of the twelve sextactic points; and it is probable that this equation would lead to various theorems connected with conics and other curves passing through these points or through points connected with them, but I shall not pursue the subject further at present.

5. By properly determining the position of α, every quartic can be expressed in the form

$$SS'=\alpha^2 w \ldots\ldots\ldots\ldots\ldots\ldots(23),$$

where S, S' are conics and α, w straight lines; from which it follows that:—*If a straight line cut a quartic in P, Q, R and S;*

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and if a conic can be described osculating the quartic at P and Q, another conic can be drawn osculating the quartic at R and S. Also the four remaining points of intersection of the conics and the quartic lie on a straight line. If α is a tangent, one of the conics must be a sextactic conic, from which it follows that:—*A conic can be drawn which osculates a quartic at the points where the tangent at a sextactic point intersects the curve.* If one of the conics breaks up into a pair of straight lines, two of the points will be points of inflexion unless they coincide, in which case the coincident point is a double point. In the latter case the direction of α is determined by the following construction:—

If through the node A a line be drawn which passes through a point of inflexion of the polar cubic of A and cuts the quartic in C and C', then a conic can be described which osculates the quartic at C and C'.

Let BC be the tangent of the quartic at C; and choose AB so that it is the harmonic conjugate of AC and the nodal tangents; then the equation of the quartic may be written

$$\alpha^2(m\beta^2+n\gamma^2)+\alpha(P\beta^3+Q\beta^2\gamma+R\beta\gamma^2+S\gamma^3)+\beta^2(\lambda\beta^2+\mu\beta\gamma+\nu\gamma^2)=0\ldots\ldots\ldots(24).$$

Let $\lambda=\lambda_1+\lambda_2$, $P=P_1+P_2$, then (24) becomes

$$\beta^2(P_1\alpha+\lambda_1\beta+\mu\gamma)+\lambda_2\beta^4+\nu\beta^2\gamma^2+\alpha\gamma^2(n\alpha+R\beta+S\gamma)+\alpha\beta^2(m\alpha+P_2\beta+Q\gamma)=0\ldots\ldots(25),$$

from which it follows that

$$u=m\alpha+R\beta+S\gamma=0\ldots\ldots\ldots\ldots\ldots(26)$$

is the tangent at the remaining point C' where CA cuts the quartic.

Since λ_2 and P_2 are arbitrary, we may take $\lambda_2=\nu m/n$, $P_2=Rm/n$; hence, *provided* $Q/m=S/n=k$ (say), (25) becomes

$$(m\beta^2+n\gamma^2)(\nu\beta^2+\alpha u)+n\beta^2\omega=0,$$

which shows that the conic $\nu\beta^2+\alpha u=0$ osculates the quartic at C and C'.

Now the polar cubic of A is

$$(m\beta^2+n\gamma^2)(2n\alpha+R\beta+kn\gamma)+P_1n\beta^2=0,$$

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which shows that the point $2\alpha + k\gamma = 0$, $\beta = 0$ is a point of inflexion. This is therefore the interpretation of the relation $Q/m = S/n$; and since a nodal cubic possesses three points of inflexion, three osculating conics can be drawn whose chords of contact pass through the node.

When the quartic has a cusp at A, we may take $\gamma = 0$ as the cuspidal tangent; and if this line be projected to infinity the quartic may be written in the form $S + \beta^3 w = 0$, where the line at infinity is the cuspidal tangent and S is the satellite conic with respect to the line β. If A is a biflecnode, any chord through A is the chord of contact of doubly osculating conic.

6. In certain cases it is possible to express a quartic in the form $SS' = \alpha^4$, in which case we have similar theorems with respect to conics which have a contact of the third order with a quartic at two points. If the line α is a tangent, the point of contact is an *octactic* point, and it is easy to determine the constants so that an anautotomic quartic should have such a point; but when the quartic is autotomic this cannot always be done. If one of the conics break up into a pair of straight lines, the line α must pass through a pair of points of undulation or a biflecnode. In the latter case we can show that:—

Through a biflecnode four straight lines can be drawn such that a conic can be described having a contact of the third order at the points where the line intersects the quartic.

Let AB, AC be the tangents at the biflecnode A; and let BC be the harmonic polar, then the equation of the quartic is

$$\alpha^2\beta\gamma = u_4$$

$$= (m\beta + n\gamma)^4 + \beta\gamma v_2,$$

whence $$\beta\gamma\,(\alpha^2 - v_2) = (m\beta + n\gamma)^4,$$

showing that the conic $\alpha^2 = v_2$ has a contact of the third order with the quartic at the points where $m\beta + n\gamma = 0$ cuts it. If P, Q be the coefficients of β^4, γ^4 in u_4, we must have $m^4/n^4 = P/Q$, which is a quartic equation for determining m/n, two of whose roots must be imaginary.

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HIGH PRIMES, $p=4\varpi+1$, $6\varpi+1$, AND FACTORISATIONS.

By Lt.-Col. ALLAN CUNNINGHAM, R.E., Fellow of King's Coll., London.

[The author is indebted to Mr. H. J. Woodall, A.R.C.Sc., for reading the proof of this Paper, and for an interesting generalisation of the Theorems of Art. 11.]

1. *High Primes.* THE large Factor Tables (Burckhardt's, Glaisher's, Dase's) show all primes <9 million. Primes >9 million require special, and usually pretty laborious, methods of detection: they may therefore fairly be styled High Primes.

2. *Primes* $p=(y^2+1)$, $\frac{1}{2}(y^2+1)$, $(y^2\pm y+1)$. The three Tables (I., II., III.) following show the whole of the High Primes (>9 million) of the above three forms under certain limits, as follows:—

57 primes, $p=\frac{1}{2}(y^2+1)$, [y *odd*], $<12\frac{1}{2}$ million,

169 primes, $p=(y^2+1)$, [y *even*], <25 million,

143 primes, $p=(y^2\pm y+1)$, <16 million.

3. *Mode of detection.* They were all detected as *primes* in the course of factorisation of numbers (N) of the two forms

$$N=y^2+1, \text{ and } N=(y^3\mp 1)+(y\mp 1)\ldots\ldots\ldots(1),$$

as follows.

4. *Factorisation of* $N=(y^2+1)$. The whole of the numbers (N) of form $N=(y^2+1)$ were factorised up to the limit $N\not>9$ million (given by $y=3000$) by aid of the large Factor-Tables.

Suppose that a certain prime (p_r) was thus found to be a factor of some one of these numbers, say of $N_r=y_r^2+1$. Then, if—as is usual—*$y_r<p_r$,

y_r, (p_r-y_r) are the two roots, (each $<p_r$), of $y_r^2+1\equiv 0$, (mod. p_r)..(2).

In this way a Table of solutions of the congruences

$$y_1^2+1\equiv 0,\ (\text{mod. } p_1);\ (y_2^2+1)\equiv 0,\ (\text{mod. } p_2);\ \ldots,\ \&c.$$

was formed for a great many primes (p_1, p_2, ..., &c.).

Each such congruence, say that of p_r, gives a factor (p_r) of *all* numbers of the general form $N_r=Y_r^2+1$, where Y_r has

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* If $y_r>p_r$, then a multiple of p_r may be subtracted, yielding a new root $(y_r-mp_r)<p_r$.

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the general form $(mp_r \mp y_r)$; thus enabling more numbers (N) to be factorised, and thereby giving rise to new congruence-solutions of above type. This process was continued up to the limit $N=(5000^2+1)$, giving therewith the solutions of the congruences $y^2+1 \equiv 0 \pmod{p}$ for *all* primes, of form $p=(4\varpi+1)$, up to* $p \not> 10{,}000$, and also (incidentally) for many higher primes.

5. *High primes* $p=\frac{1}{2}(y^2+1)$, and (y^2+1). It will now be clear that *all* the numbers (N or $\frac{1}{2}N$) above dealt with, for which no factor $< \sqrt{N}$ could be found, must necessarily be *prime*. Thus were found the 57 High Primes $p=\frac{1}{2}(y^2+1)$, [*y odd*], and the 169 High Primes $p=(y^2+1)$, [*y even*], of the Tables I., II. below.

6. *Residuacity*† *of* 2 *to above.* The above primes are all of form $p=4\varpi+1$, and may be expressed as a sum of squares as

i. *y even*, $=2\eta$; $p=1^2+y^2=1^2+(2\eta)^2$ .....(3*a*),

ii. *y odd*, $=4\eta \mp 1$; $p=\frac{1}{2}(1+y^2)=\left(\frac{y-1}{2}\right)^2+\left(\frac{y+1}{2}\right)^2$

$=(2\eta \mp 1)^2+(2\eta)^2$ .....(3*b*).

The criteria that 2 should be a 2ic, 4ic, 8ic Residue may now be briefly expressed; (using ω, ε to denote an *odd* or *even* number respectively):

$(2/p)_2=-1$, if $\eta=\omega$; $(2/p)_2=+1$, and $(2/p)_4=-1$, if $\eta=2\omega$

.....(4*a*),

$(2/p)_4=+1$, and $(2/p)_8=-1$, if $\eta=4\omega$; $(2/p)_8=+1$, if $\eta=8\omega$

.....(4*b*).

7. *Properties of* $N=y^2+y+1$. Here, writing $y'=y+1$,

$N=y^2+y+1=(y^3-1)\div(y-1)$.....(5*a*)

$=y'^2-y'+1=(y'^3+1)\div(y'+1)$.....(5*b*)

$=(1+\frac{1}{2}y)^2+3(\frac{1}{2}y)^2$, if y is *even*, y' *odd*.....(6*a*)

$=(1-\frac{1}{2}y')^2+3(\frac{1}{2}y')^2$, if y is *odd*, y' *even* .....(6*b*).

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* Reuschle gives a similar Table showing the *least* root of $y^2+1 \equiv 0$, (mod. p), up to $p \not> 1000$ only, at p. 441 of *Tafeln complexer Primzahlen*, Berlin, 1875.

† A term introduced by the writer to denote "residual character."

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Thus any number N, expressed in any *one* of the four forms $(5a, b)$ is at once expressible in *each* of them, and also in *one* of the forms $(6a, b)$.

8. *Factorisation of* $N=y^2+y+1$. The process used was quite similar to that of Art. 4. The whole of the numbers (N) of form $N=(y^2+y+1)$ were factorised up to the limit $N \not> 9$ million, (given by $y=3000$), by aid of the large Factor-Tables.

Supposing that a certain prime (p_r) was thus found to be a factor of some one of these numbers, say of $N_r=y_r^2+y_r+1$: then, if—as is usual—*$y_r < p_r$; and writing $y_r'=y_r+1$,

y_r, (p_r-y_r') are the two roots, (each $<p_r$), of $y_r^2+y_r+1\equiv 0$, (mod. p_r) .....(7a),

y_r', (p_r-y_r) are the two roots, (each $<p_r$), of $y_r'^2-y_r'+1\equiv 0$, (mod. p_r) .....(7b).

In this way a Table of solution of congruences of the above type was formed for a great many primes $(p_1, p_2, \&c.)$. With their aid the process was continued, as explained in Art. 4, up to the limit $N=(4000^2+4000+1)$, giving therewith the roots of the congruences of above type for *all* primes of form $p=6\varpi+1$ up to $p \not> 8000$, and with some omissions up to $p \not> 10{,}000$, and also (incidentally) for many higher primes. The solutions of the few cases missing under 10,000 were supplied by special means (which need not be detailed here), so that the congruence-solution Table has been completed up to p† $\not> 10{,}000$.

9. *High Primes,* $p=y^2+y+1$. It will now be clear that *all* the numbers (N) dealt with as in Art. 8, for which no factor $<\sqrt{N}$ could be found must necessarily be *prime*. Thus were found the 143 High Primes, $p=(y^2+y+1)$ of Table III. at the end.

10. *Extended factorisations.* The two congruence-solution‡ Tables above-mentioned being *complete* for all primes of forms

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* If $y_r > p_r$, a new root $(y_r-mp_r) < p_r$ may be found by subtracting a multiple of p_r.

† Reuschle gives a similar Table showing the roots $(<\frac{1}{2}p)$ of $y^2\pm y+1\equiv 0$, (mod. p), up to $p \not> 1000$ only at p. 1 of his work above quoted.

‡ It is hoped to publish these congruence-solution Tables, together with many other similar ones compiled by the present author, and now far advanced, on some future occasion.

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$p = 4\varpi + 1$, $6\varpi + 1 \not> 10{,}000$, suffice for the complete factorisation of *all* numbers (N) of forms

$$N = (y^2 + 1),\ (y^2 + y + 1),\ (y'^2 - y' + 1) \not> 100 \text{ million},$$

and for many higher ones of these forms, and therefore also for the detection of *all** (lesser) primes therein contained); *e.g.* all primes of form

$$p = \tfrac{1}{2}(y^2 + 1),\ p = \tfrac{1}{3}(y^2 + y + 1),\ p = \tfrac{1}{3}(y'^2 - y' + 1) \not> 100 \text{ mills}.$$

11. *High factorisable forms.* Let y be the *lesser* root of any of the above congruences; then (writing $y' = y + 1$)

$Y = p - y,\ p + y$ also satisfy $Y^2 + 1 \equiv 0$, (mod. p) .....(8a),

$Y = (p - y'),\ (p + y)$ also satisfy $Y^2 + Y + 1 \equiv 0$, (mod. p)...(8b).

Now, let N_y denote a number of one of the forms above discussed, and formed from the base y.* It will be shown that—

i. When $N_y = y^2 + 1 = p$,

then $N_{p-y} = N_y . N_{y-1},\ N_{p+y} = N_y . N_{y+1}$ .....(9a).

ii. When $\frac{1}{2}N_y = \frac{1}{2}(y^2 + 1) = p$,

then $N_{p-y} = \frac{1}{4}N_y . N_{y-2},\ N_{p+y} = \frac{1}{4}N_y . N_{y+2}$ .....(9b).

iii. When $N_y = y^2 + y + 1 = p$,

then $N_{p-y'} = N_y . N_{y-1},\ N_{p+y} = N_y . N_{y+1}$ .....(10a).

iv. When $\frac{1}{3}N_y = \frac{1}{3}(y^2 + y + 1) = p$,

then $N_{p-y'} = \frac{1}{9}N_y . N_{y-2},\ N_{p+y} = \frac{1}{9}N_y . N_{y+2}$ .....(10b).

The proof is as follows:—

Case i. When $N_y = y^2 + 1 = p$,

$$N_{p \mp y} = (p \mp y)^2 + 1 = p^2 \mp 2py + p = p.(p \mp 2y + 1)$$
$$= p.(y^2 + 1 \mp 2y + 1) = p.\{(y \mp 1)^2 + 1\} = N_y . N_{y \mp 1} \ldots(9a).$$

Case ii. When $\frac{1}{2}N_y = \frac{1}{2}(y^2 + 1) = p$,

$$N_{p \mp y} = (p \mp y)^2 + 1 = p^2 \mp 2py + 2p = p.(p \mp 2y + 2)$$
$$= p.\left(\frac{y^2 + 1}{2} \mp 2y + 2\right) = p.\frac{(y \mp 2)^2 + 1}{2} = \tfrac{1}{4}N_y . N_{y \mp 2} \ldots(9b).$$

* These primes could be discovered without the actual factorisation of the numbers N, and therefore with much less labor than before; this work is being undertaken, and is now far advanced.

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Case iii. When $N_y = y^2+y+1 = y'^2-y'+1 = p$, $[y'=y+1]$,

$$N_{p-y'} = (p-y')^2+(p-y')+1 = p^2-2py'+2p = p.(p-2y'+2)$$
$$= p.(y^2+y+1-2y) = p.\{(y-1)^2+(y-1)+1\} = N_y.N_{y-1} \ldots (10a),$$

$$N_{p+y} = (p+y)^2+(p+y)+1 = p^2+2py+2p = p(p+2y+2)$$
$$= p.(y^2+y+1+2y+2) = p.\{(y+1)^2+(y+1)+1\} = N_y.N_{y+1} \ldots (10a).$$

Case iv. When $\frac{1}{3}N_y = \frac{1}{3}(y^2+y+1) = \frac{1}{3}(y'^2-y'+1) = p$, $[y'=y+1]$,

$$N_{p-y'} = (p-y')^2+(p-y')+1 = p^2-2py'+4p = p(p-2y'+4)$$
$$= p.\left(\frac{y^2+y+1}{3}-2y+2\right) = p.\frac{y^2-5y+7}{3}$$
$$= p.\frac{(y-3)^2+(y-3)+1}{3} = \tfrac{1}{3}N_y.N_{y-3} \quad \ldots\ldots\ldots\ldots\ldots (10b),$$

$$N_{p+y} = (p+y)^2+(p+y)+1 = p^2+2py+4p = p(p+2y+4)$$
$$= p.\left(\frac{y^2+y+1}{3}+2y+4\right) = p.\frac{y^2+7y+13}{3}$$
$$= p.\frac{(y+3)^2+(y+3)+1}{3} = \tfrac{1}{3}N_y.N_{y+3} \quad \ldots\ldots\ldots\ldots\ldots (10b).$$

Thus the Theorems i–iv above are all proved. The numbers N, hereby factorisable, are very large, being of the order y^4, since N_y, $N_{y\mp1}$, $N_{y\mp3}$, are of the order y^2; and they are *completely* factorisable when the latter numbers can be factorised.

11*a*. *Examples.* The factorisation-tables above-mentioned as *complete* to the limits—

$$y = 5000 \text{ for } N = y^2+1; \quad y = 4000 \text{ for } N = y^2+y+1$$

give the *immediate* (complete) factorisation of *all* numbers N_Y of the types above treated, where $Y = p \mp y$, or $p-y'$, and N_y, $\frac{1}{3}N_y$, or $\frac{1}{3}N_y = p$, up to the above limits of y. Some of the most interesting examples occur when the smaller factor $N_{y\mp1}$, $\frac{1}{2}N_{y\mp2}$, $\frac{1}{3}N_{y\mp3}$ is also *prime*; (see Tables I., II., III.).

Ex. of i. $y = 4886$ gives $N_y = p$; $y-1 = 4885$ gives $\frac{1}{2}N_{y-1} = p'$,

$$\therefore N_{p\ y} = 23868111^2+1 = 2p'p = 2.11931613.23372997.$$

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Again $y=4904$ gives $N_y=p$; $y+1=4905$ gives $\frac{1}{2}N_{y+1}=p'$,

$$\therefore N_{y+y}=24054121^2+1=2p'p=2.12029513.24049217.$$

Ex. of ii. $y=4811$ gives $\frac{1}{2}N_y=p$; $y-2=4809$ gives $\frac{1}{2}N_{y-2}=p'$,

$$\therefore N_{p.y}=11568050^2+1=p'p=11563241.11572861.$$

Ex. of iii. $y=3996$ gives $N_y=p$; $y-1=3995$ gives $N_{y-1}=p'$,

$$\therefore N_{p.y}=(15968016^2-1)\div(15968016-1)=p'p=15964021.15972013.$$

12. *Higher factorisable forms.* Interesting results leading to still higher factorisable numbers are obtained by multiplying together the *pairs* of results in the formulæ i, ii, iii, iv respectively of Art. 11.

Cases i. and ii. $N_y=y^2+1$. Take $\mathbf{N}=N_{p-y}.N_{p+y}$,

therefore $\mathbf{N}=\{(p-y)^2+1^2\}.\{(p+y)^2+1^2\}$

$$=(p^2-y^2+1)^2+(2y)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(11),$$

and, by Art. 11,

i. If $N_y=y^2+1=p$, [*y even*],

$$\mathbf{N}=(y^4+y^2+2)^2+(2y)^2=N_{y-1}.N_y.N_y.N_{y+1} \quad \ldots\ldots(11a).$$

ii. If $\frac{1}{2}N_y=\frac{1}{2}(y^2+1)=p$, [*y odd*],

$$\mathbf{N}=\tfrac{1}{16}.(y^4-2y^2+5)^2+(2y)^2=\tfrac{1}{16}N_{y-2}.N_y.N_y.N_{y+2} \quad \ldots(11b).$$

12*a*. *Example of i.* Interesting examples occur when $\frac{1}{2}N_{y-1}$, N_y, $\frac{1}{2}N_{y+1}$ are all *prime*, e.g. $y=4300$; see Tables I., II,

$$\mathbf{N}=(4300^4+4300^2+2)^2+8600^2=4.9240701.18490001^2.9249301.$$

Example of ii. (*y odd*). Here $\frac{1}{2}N_{y-2}$, $\frac{1}{2}N_y$, $\frac{1}{2}N_{y+2}$ cannot *all* be prime. But interesting examples occur when $\frac{1}{2}N_{y-2}$, $\frac{1}{2}N_{y+2}$ are *both prime*, e.g. $y=4903$; see Table I.,

$$\mathbf{N}=\tfrac{1}{16}(4903^4-2.4903^2+5)^2+8906^2=12009901.(5.2403941)^2.12029513.$$

13. *Bin-Aurifeuillians.* These are numbers* **N** of form below, *algebraically* resolvable into two factors L, M.

$$\mathbf{N}=x^4+4y^4=L.M \quad \ldots\ldots\ldots\ldots\ldots(12),$$

where $L=x^2-2xy+2y^2=(x-y)^2+y^2$,

and $M=x^2+2xy+2y^2=(x+y)^2+y^2 \quad \ldots\ldots\ldots(12a).$

* These numbers are treated of at some length in the author's paper "On Aurifeuillians," in *Proc. Lond. Math. Soc.*, Vol. XXIX.

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Writing $x = 1$ gives the simpler form

$$\mathbf{N} = 1 + 4y^4 = L.M \quad \ldots\ldots\ldots\ldots\ldots\ldots(13),$$

$$L = 1 - 2y + 2y^2 = (y-1)^2 + y^2,$$

$$M = 1 + 2y + 2y^2 = y^2 + (y+1)^2 \quad \ldots\ldots\ldots\ldots(13a).$$

Examples. The numbers **N** of this latter form are *all* (completely) factorisable by the large Factor-Tables up to the limit

$$\mathbf{N} = 1 + 4.2121^4 = (2120^2 + 2121^2)(2121^2 + 2122^2)$$
$$= 8993041 . (25 . 13 . 27699);$$

and are further *all* (completely) factorisable by aid of the Congruence-Table mentioned in Art. 4 up to the limit

$$\mathbf{N} = 1 + 4.7071^4 = (7070^2 + 7071^2)(7071^2 + 7072^2)$$
$$= (61 . 1639081)(25 . 4000489);$$

and, in many cases to much higher limits, as *e. g.*

$$\mathbf{N} = {}^*1 + 4.10^{16} = (1 - 2.10^4 + 2.10^8)(1 + 2.10^4 + 2.10^8)$$
$$= (13 . 41 . 457 . 821)(569 . 351529);$$

$$\mathbf{N} = {}^*1 + 4.10^{20} = (1 - 2.10^5 + 2.10^{10})(1 + 2.10^5 + 2.10^{10})$$
$$= (53 . 5953 . 63389)(17 . 29 . 1129 . 35933).$$

13*a*. Again, writing $N = y^2 + 1$, as before,

$$\mathbf{N} = y^4 + 4.1^4 = \{(y-1)^2 + 1^2\} . \{(y+1)^2 + 1^2\} = N_{y-1} . N_{y+1} \quad \ldots(14).$$

Thus the factorisation-table of N_y, mentioned in Art. 4 as *complete* to $y = 5000$, gives the *immediate* (complete) factorisation of *all* such numbers **N** to same limit of y.

Ex. Interesting cases occur when N_{y-1}, N_{y+1} are both primes, or doubles of primes, *e. g.*

(1) Take $y = 4755$ (and see Table II.),

$$\mathbf{N} = 4955^4 + 4 = (4954^2 + 1^2)(4956^2 + 1^2) = 24542117 . 24561937;$$

(2) Take $y = 4810$ (and see Table I.),

$$\tfrac{1}{4} . \mathbf{N} = \tfrac{1}{4}(4810^4 + 4) = \tfrac{1}{2}(4809^2 + 1) . \tfrac{1}{2}(4811^2 + 1) = 11563241 . 11572861;$$

the same result as obtained otherwise in Art. 11*a*.

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* The factorisation of numbers of this kind (involving powers of 10) is greatly facilitated by the help of a Table of Residues of powers of 10; the author has compiled such a Table showing the Residue of 10^x, (mod. p) from $x = 1$ to 12, for all prime, and powers of prime, moduli > 10000.

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14. *Trin-Aurifeuillians.* These are numbers* N of form below, *algebraically* resolvable into two factors L, M,

$$\mathrm{N} = (x^6 + 3^3y^6) \div (x^2 + 3y^2) = x^4 - 3x^2y^2 + 9y^4 = L.M \ldots\ldots (15),$$

where $L = x^2 - 3xy + 3y^2$, $M = x^2 + 3xy + 3y^2 \ldots\ldots (15a)$.

Writing $x = 1$ gives the simple form

$$\mathrm{N} = (1 + 3^3y^6) \div (1 + 3y^2) = 1 - 3y^2 + 9y^4 = L.M \ldots\ldots\ldots (16),$$

where $L = 1 - 3y + 3y^2$, $M = 1 + 2y + 3y^2 \ldots\ldots\ldots (16a)$.

Examples. The numbers N of the latter form are *all* (completely) factorisable by the large Factor-Tables up to the limit

$$\mathrm{N} = (1 + 3^3.1731^6) \div (1 + 3.1731^2)$$

$$= (1 - 3.1731 + 3.1731^2)(1 + 3.1731 + 3.1731^2) = (7.73.17581)(19.473383);$$

and are further *all* (completely) factorisable with the aid of the Congruence-Table mentioned in Art. 8 up to the limit

$$\mathrm{N} = (1 + 3^3.5773^6) \div (1 + 3.5773^2)$$

$$= (1 - 3.5773 + 3.5773^2)(1 + 3.5773 + 3.5773^2) = (739.135271)(7.19.751879);$$

and, in many cases, to very much higher limits, as *e.g.*

$$\mathrm{N}\dagger = (1 + 3^3.10^{24}) \div (1 + 3.10^8) = (1 - 3.10^4 + 3.10^8)(1 + 3.10^4 + 3.10^8)$$

$$= (61.1543.3187)(300030001\ddagger).$$

14*a*. Again, writing $N_y = y^2 + y + 1$, as before,

$$\mathrm{N} = (y^6 + 3^3) \div (y^2 + 3) = (y^4 - 3y^2 + 9) \ldots\ldots\ldots\ldots (17)$$

$$= (y^2 - 3y + 3)(y^2 + 3y + 3) \ldots\ldots\ldots\ldots (17a)$$

$$= \{(y-1)^2 - (y-1) + 1\}\{(y+1)^2 + (y+1) + 1\} \ldots (17b)$$

$$= \{(y-2)^2 + (y-2) + 1\}.\{(y+1)^2 + (y+1) + 1\} = N_{y-2}.N_{y+1} \ldots (17c).$$

Examples. The factorisation-table of N_y, mentioned in Art. 8 as *complete* to $y = 4000$, gives the *immediate* (complete) factorisation of *all* such numbers N to same limit of y.

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* These numbers are treated of at some length in the author's paper "On Aurifeuillians," in *Proc. Lond. Math. Soc.*, Vol. XXIX.

† The factorisation of numbers of this kind (involving powers of 10) is much facilitated by a Table of Residues of powers of 10, as explained in the footnote of (Art. 18).

‡ This number is shown by the congruence-table to contain no divisor < 10000; but, being $> 10000^2$, is beyond the powers of the congruence-table to resolve.

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Ex. Interesting cases occur when N_{y-3}, N_{y+1} are both prime, *e. g.* Take $y = 3988$, and see Table III.,

$$N = (3988^3 + 3^3) \div (3988^2 + 3) = 15892183 . 15916111.$$

Take $y = 3996$, and see Table III.,

$$N = (3996^3 + 3^3) \div (3996^2 + 3) = 15964021 . 15988003.$$

15. *Arithmetical Work.* The whole of the factorisation described in Arts. 4, 8 was done for the author, and under his superintendence, by *one* Assistant (Miss B. B. Haselden); every factorisation was tested by *casting out the nines.* Every number which had escaped factorisation by this Assistant was then examined throughout the whole range of the Congruence-tables, so as to find its factors (if any), by *two* other Assistants, *working independently* (Miss B. E. Haselden and Miss A. Woodward). Thus no number was accepted as a *High Prime* until *three* independent computers had failed to find factors.

16. *Addendum to Art.* 11. The theorems (i–iv) of Art. 11 may be generalised* as follows; for facility of comparison the same reference numbers are attached to the generalisations as to the originals.

i. If $N_y = y^3 + a = p$; then $N_{p \mp y} = N_y . N_{y \mp 1}$ .....(9a').

ii. If $\frac{1}{b} . N_y = \frac{1}{b} . (y^3 + a) = p$;

then $$N_{p \mp y} = \frac{1}{b^2} . N_y . N_{y \mp b} .....(9b').$$

iii. If $N_y = y^2 + ay + a^2 = y'^2 - ay' + a^2 = p$, $[y' = y + a]$;

then $$N_{p-y'} = N_y . N_{y-1}\ ;\ N_{p+y} = N_y . N_{y+1} .....(10a').$$

iv. If $\frac{1}{b} . N_y = \frac{1}{b^2}(y^2 + ay + a^2) = \frac{1}{b} . (y'^2 - ay' + a^2) = p$, $[y' = y + a]$;

then $$N_{p-y'} = \frac{1}{b^2} . N_y . N_{y-b},\quad N_{p+y} = \frac{1}{b^2} . N_y . N_{y+b} .....(10b').$$

The proof of these theorems is quite similar to that of the originals (Art. 11); and they merge into the originals by writing $a = 1$, $b = 1$.

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* This generalisation is due to Mr. H. J. Woodall, A.R.C.Sc.

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57 *High Primes*, $p = \frac{1}{2}(y^3+1)$. Table I.

p	y	p	y	p	y	p	y
9,		10,		11,		12,	
095113	4265	030721	4479	002741	4691	009901	4901
197761	4289	057613	4485	087341	4709	029513	4905
240701	4299	102513	4495	134481	4719	206741	4941
249301	4301	174561	4511	143921	4721	246301	4949
309613	4315	264981	4531	324041	4759	325613	4965
439513	4345	401361	4561	381221	4771	395221	4979
456901	4349	511113	4585	496013	4795	495001	4999
465601	4351	529461	4589	515201	4799		
500441	4359	621441	4609	524801	4801		
509161	4361	649113	4615	563241	4809		
552821	4371	741613	4635	572861	4811		
570313	4375	760161	4639	640313	4825		
834613	4435	815901	4651	804941	4859		
852361	4439	862461	4661	834113	4865		
879013	4445	899781	4669	863321	4871		
941341	4459	993361	4689	902321	4879		
968113	4465			931613	4885		

169 *High Primes*, $p = y^3+1$. Table II.

p	y	p	y	p	y	p	y
9,		10,		11,		12,	
096257	3016	074277	3174	062277	3326	068677	3474
156677	3026	137857	3184	115557	3334	110401	3480
278117	3046	214417	3196	155601	3340	180101	3490
326917	3054	265617	3204	222501	3350	278017	3504
449477	3074	329797	3214	262737	3356	362257	3516
572837	3094	368401	3220	289601	3360	390401	3520
647237	3106	497601	3240	383877	3374	460901	3530
672101	3110	536517	3246	492101	3390	489157	3534
821957	3134	588517	3254	806097	3436	503297	3536
834497	3136	666757	3266	874917	3446		
859601	3140	719077	3274				
960337	3156	758401	3280				
985601	3160	824101	3290				
		916417	3304				
		929637	3306				
		982597	3314				

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High Primes, $p = y^2 + 1$. Table II. (*continued*).

p	y	p	y	p	y	p	y
13,		14,		15,		16,	
133377	3624	032517	3746	085457	3884	048037	4006
278737	3644	092517	3754	132101	3890	192577	4024
322501	3650	107537	3756	163237	3894	208677	4026
395601	3660	167697	3764	210001	3900	273157	4034
468901	3670	243077	3774	288101	3910	370117	4046
586597	3686	258177	3776	397777	3924	451137	4056
808657	3716	318657	3784	570917	3946	564901	4070
912901	3730	364101	3790	729157	3966	646401	4080
942757	3734	394437	3794	872257	3984	695397	4086
		440001	3800	952037	3994	924997	4114
		485637	3806			974401	4120
		638277	3826				
		822501	3850				
		976901	3870				
17,		18,		19,		20,	
007377	4124	062501	4250	044497	4364	016677	4474
106497	4136	147601	4260	061957	4366	124197	4486
139601	4140	198757	4266	096901	4370	214017	4496
189317	4146	438437	4294	131877	4374	286017	4504
255717	4154	490001	4300	219457	4384	340101	4510
272337	4156	576101	4310	395217	4404	466577	4524
388901	4170	748901	4330	448101	4410	520901	4530
422277	4174	800897	4336	483397	4414	557157	4534
438977	4176	835601	4340	749137	4444	611601	4540
472401	4180			855937	4456	738917	4554
505857	4184					848357	4566
690437	4206						
859077	4226						
21,		22,		23,		24,	
068101	4590	090001	4700	001617	4796	049217	4904
160001	4600	127617	4704	367557	4834	108101	4910
196817	4604	240657	4716	522501	4850	206401	4920
215237	4606	335077	4726	775377	4876	364097	4936
288997	4614	410757	4734	872997	4886	443137	4944
307457	4616	429697	4736	951237	4894	542117	4954
566737	4644	600517	4754			561937	4956
622501	4650	848401	4780			900101	4990
771557	4666	886657	4784				
		905797	4786				
		982437	4794				

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143 *High Primes*, $p = (y^s - 1) \div (y - 1)$. Table III.

p	y	p	y	p	y	p	y
9,		10,		12,		14,	
069133	3011	696171	3270	197557	3492	021281	3744
123421	3020	768243	3281	260503	3501	036263	3746
147601	3024	814233	3288	323611	3510	156407	3762
165757	3027	866913	3296	358741	3515	224213	3771
183931	3030	873507	3297	492691	3534	330011	3785
238561	3039	906507	3302	535141	3540	398231	3794
250723	3041	985911	3314	549307	3542	451403	3801
293353	3048			556393	3543	657413	3828
305551	3050	11,		598951	3549	703391	3834
323863	3053	092231	3330	634471	3554	718733	3836
421831	3069	145583	3338	655807	3557	741761	3839
569743	3093	152261	3339	684283	3561	764807	3842
606901	3099	192371	3345	698533	3563	772493	3843
619303	3101	427781	3380	769903	3573	795563	3846
656557	3107	434543	3381	791353	3576	864881	3855
700111	3114	529421	3395	870157	3587	996257	3872
787513	3128	549803	3398	920431	3594		
831361	3135	597431	3405	985213	3603	15,	
843907	3137	611057	3407			050521	3879
881593	3143	679307	3417	13,		066043	3881
887881	3144	692981	3419	006843	3606	089341	3884
944563	3153	720353	3423	021273	3608	097111	3885
		816407	3437	042933	3611	120433	3888
10,		823283	3438	151503	3626	143773	3891
052071	3170	857693	3443	173271	3629	206101	3899
077451	3174	864581	3444	238683	3638	252931	3905
211221	3195	899051	3449	245961	3639	323311	3914
224007	3197	961223	3458	289671	3645	354643	3918
243201	3200	968141	3459	355371	3654	370321	3920
268821	3204			479913	3671	425257	3927
320157	3212	12,		597657	3687	519661	3939
326583	3213	002761	3464	634557	3692	638071	3954
378063	3221	044371	3470	775233	3711	725191	3965
461991	3234	065203	3473	790083	3713	804601	3975
513807	3242	106921	3479	842121	3720	892183	3986
572253	3251	127807	3482	931557	3732	916111	3989
591771	3254	148711	3485	968907	3737	964021	3995
598281	3255	190573	3491	998823	3741	972013	3996
						988003	3998

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RESEARCHES IN THE THEORY OF DIVERGENT SERIES AND DIVERGENT INTEGRALS.

By G. H. HARDY, Trinity College, Cambridge.

Introduction.

§ 1. SINCE the appearance of M. Borel's original memoirs on Divergent Series, few branches of analysis have aroused more interest among mathematicians. The applications of divergent series to the theory of functions of a complex variable, and especially to the problems of analytic continuation and the asymptotic expansion, have given rise to extensive researches; I need only refer to the writings of M. Borel himself, of MM. Servant and Le Roy, and of Mr. Barnes. The 'arithmetic' theory of divergent series, as it was originally presented by M. Borel, has received less attention.

In the first few sections of this paper I discuss a few of the questions which arise in the early part of the theory; this part of the paper may be regarded as supplementary to Chap. III. of M. Borel's *Leçons sur les Séries Divergentes*, and to my paper 'On differentiation and integration of Divergent Series.'* I discuss in turn (i) the relation of convergence and absolute summability, (ii) the removal of terms from, or addition of terms to, a divergent series, (iii) the 'condition of consistency' for definitions other than M. Borel's original definition, (iv) the relation of 'generalised limits' and 'mean values,' and (v) the multiplication of divergent series. In the latter sections I suggest the outlines of a similar theory of divergent integrals.

PART I.

SOME POINTS IN THE THEORY OF DIVERGENT SERIES.

General Preliminaries.

§ 2. In the first paragraph of the paper referred to, I indicated very briefly a general principle which may often guide us in our choice of a convention as to the value to be attributed to an otherwise meaningless expression. It is that

* *Cambridge Philosophical Transactions*, XIX., p. 297.

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when a number of limit operations, performed in a definite order on a function of several variables, lead to a definite result, but do not do so when performed in another definite order, we are to agree that the expression which is the result of the formal carrying out of the second sequence of operations MEANS *the result of the first sequence.*

I pointed out that Borel's definition of the sum of a divergent series was an application of this principle; and the same is true of the definitions given by M. Le Roy* and Mr. Barnes.† It is at first sight a little difficult to imagine how we can gain any advantage by writing X when we mean Y. Results, however, prove conclusively that we can. And if we consider an example, it is easy to see how.

Few analytical processes occur more frequently than that of the evaluation of a definite integral by expansion of the subject of integration, and integration term by term. Our work in this case depends on an equation of the form

$$(1) \quad \int_\beta^\gamma \sum_0^\infty u_n(\alpha)\, d\alpha = \sum_0^\infty \int_\beta^\gamma u_n(\alpha)\, d\alpha.$$

Now, if $\Sigma u_n(\alpha)$ and $\Sigma \int_\beta^\gamma u_n(\alpha)\, d\alpha$ are convergent,

$$\sum_0^\infty u_n(\alpha) = \int_0^\infty e^{-x} dx \sum_0^\infty \frac{x^n}{n!} u_n(\alpha)$$

and

$$\sum_0^\infty \int_\beta^\gamma u_n(\alpha)\, d\alpha = \int_0^\infty e^{-x} dx \sum_0^\infty \frac{x^n}{n!} \int_\beta^\gamma u_n(\alpha)\, d\alpha.$$

Thus the simple transformation (1) which may be represented symbolically by

$$\int_{(1)} \Sigma = \Sigma \int_{(1)}$$

may be replaced by a more elaborate permutation of limit operations, viz.

$$(2) \quad \int_{(1)} \int_{(2)} \Sigma = \int_{(2)} \int_{(1)} \Sigma = \int_{(2)} \Sigma \int_{(1)},$$

the first and last expressions being equivalent to the two members of (1).

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* *Annales de la Faculté des Sciences de Toulouse*, 1900, pp. 317–431.
† *Phil. Trans.* (A.), 199, pp. 411–500.

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Now (2) may be true when neither or only one side of (1) is determinate. In this case, if we adopt Borel's definition of the sum of a divergent series, and the notation of my former paper,* (2) may be written in the form

$$(3)\quad \int_{(1)} S = \left[\int_{(2)}\int_{(1)} \Sigma = \right] = S\int_{(1)}.$$

In practice we omit the intermediate step, so that the transformation is formally the same as that expressed by (1); and so our work gains immensely in quickness and compactness, to say nothing of the probability that but for the formal resemblance of (1) and (3) we should never have thought of the transformation (2). Thus in this, and in a greater degree in more complicated cases, divergent series provide us with a suggestive and easily manipulated shorthand representation of elaborate and often very difficult analytical processes.

They do not, however, exempt us from the labour of examining the legitimacy of these processes. Fortunately this is not very difficult in most of the cases which ordinarily occur. In the theory of divergent power series in particular, we are not troubled seriously by cases of exception. In the 'arithmetic' theory, the part of the theory which is concerned with the application of divergent series to the theory of functions of real variables, the fundamental difficulties of the subject are naturally more prominent.†

§ 3. I shall now consider one or two of the questions which arise in the beginning of the theory.

Convergence and Absolute Summability.

In my former paper I proved that what I called the 'Condition of consistency' was satisfied by M. Borel's definition; that is to say that

$$\overset{\infty}{\underset{0}{S}} u_n = \overset{\infty}{\underset{0}{\Sigma}} u_n$$

whenever the latter series is convergent. That this is so was stated by M. Borel in his first memoir, *Fondements de la théorie des séries divergentes summables*,‡ and again in his *Memoire sur les séries divergentes*§; but I am unable to find

* I use S instead of the symbol used there.
† See *e.g.*, §§ 19–26 of my paper referred to above.
‡ *J. de Math.*, 1896, pp. 103–122.
§ *Annales de l'École Normal Supérieure*, 1899, pp. 1–131.

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any proof in any of his writings. In his *Leçons sur les séries divergentes*, M. Borel makes the further assertion that all convergent series are 'evidently' *absolutely* summable; that is, that if

$$\Sigma u_n$$

is convergent, and

$$u(x) = \Sigma \frac{x^n}{n!} u_n,$$

all the integrals

$$\int_0^\infty e^{-x} \left| \frac{d^\lambda}{dx^\lambda} u(x) \right| dx \quad (\lambda \geqq 0)$$

are convergent. This, it seems to me, is untrue; unless the analysis which follows is inaccurate, there are convergent series which are *not* absolutely summable.*

Consider the series

$$0 - 1 + 0 + 0 + \tfrac{1}{2} + 0 + 0 + 0 + 0$$
$$- \tfrac{1}{3} + 0 + 0 + 0 + 0 + 0 + 0 + \tfrac{1}{4} + \ldots,$$

in which

$$u_n = \frac{(-)^{\sqrt{n}}}{\sqrt{n}},$$

if n is a square, and $= 0$ otherwise. The series is evidently convergent.

I shall prove that if

$$u(x) = \sum_1^\infty \frac{(-)^i}{i} \frac{x^{i^2}}{i^2!},$$

$$| e^{-x} u(x) | > \frac{K}{x}$$

throughout a certain infinite series of intervals. I may remark that this example was suggested to me by an investigation of M. Borel's† concerning the series

$$1 + x + x^4 + x^9 + \ldots,$$

and that I adopt a very convenient notation used by him elsewhere.‡ I denote by K a constant entering into an

* I give this investigation more because it shows how we can determine the manner of growth of a very interesting type of function, than because the question of the truth or falsity of M. Borel's statement is particularly important.

† *J. de Math.*, 1896, p. 447.

‡ *Comptes Rendus*, 11 Mai 1896.

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equality or inequality, not necessarily the same in all inequalities, but always lying between, say, 0·00001 and 10000. Also when I say that one quantity is approximately another, I mean that their ratio differs from unity by less than σ, a very small positive quantity fixed throughout the investigation.

We can choose I so that

$$i^{i^2}! = i^{2i^2+1} e^{-i^2} \sqrt{(2\pi)}\,(1+\rho),$$

$$|\rho| < \sigma,$$

for all values of $i > I$. When this has been done we may evidently neglect the terms of $u(x)$ up to $i = I$, as for sufficiently large values of x,

$$e^{-x} \left| \sum_1^I \frac{(-)^i}{i} \frac{x^{i^2}}{i^2!} \right|$$

is less than any power of $\frac{1}{x}$.

Let
$$v_i = \frac{1}{i} \frac{x^{i^2}}{i^2!},$$

$$[\sqrt{x}] = X,$$

$$\sqrt{x} = X + f,$$

and suppose $0 < f < \frac{1}{4}$. Then it can be shown that $|u(x)|$ is of the same order of greatness as its X^{th} term.

In the first place, neglecting factors which are approximately unity,

$$\frac{v_i}{v_{i+1}} = (ex)^{-2i-1} \frac{(i+1)^{2i^2+4i+2}}{i^{2i^2+1}}$$

$$= (ex)^{-2i-1} (i+1)^{4i+2} \left(1+\frac{1}{i}\right)^{2i^2+i}$$

$$= e^{-x} \left(\frac{i+1}{X+f}\right)^{4i+2},$$

as
$$\left(1+\frac{1}{i}\right)^{2i^2} = e^{2i^2 \log\left(1+\frac{1}{i}\right)}$$

$$= e^{2i-1+\frac{1}{3i}-\cdots}.$$

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Hence if $i \leqq X-1$,

$$\frac{v_i}{v_{i+1}} < \tfrac{1}{4}.$$

Hence

$$\sum_{I+1}^{X} (-)^i v_i = (-)^X \{(v_X - v_{X-1}) + (v_{X-2} - v_{X-3}) + \ldots \pm v_{I+1}\}$$

has the sign of $(-)^X$, and is numerically greater than

$$\tfrac{3}{4} v_X.$$

If $i = X$,

$$\frac{v_X}{v_{X+1}} = e^{-1}\left(1 + \frac{1-f}{X+f}\right)^{4X+2}$$
$$= e^{2-4f}$$

approximately, *i.e.* $> e > 2$.

If $i > X$ it is evident that the ratio of v_i to v_{i+1} is greater still. Hence

$$\sum_{X+1}^{\infty} (-)^i v_i = (-)^{X+1} \{v_{X+1} - (v_{X+2} - v_{X+3}) - \ldots\}$$

is numerically less than $\frac{1}{2} v_X$.

Hence $$\left| \sum_{I+1}^{\infty} (-)^i v_i \right| > \tfrac{1}{4} v_X.$$

Now v_X is approximately

$$\frac{1}{X} \frac{(ex)^{X^2}}{X^{2X^2+1}\sqrt{(2\pi)}}$$
$$= \frac{K}{X^3} e^{X^2} \left(1 + \frac{f}{X}\right)^{2X^2}$$
$$= \frac{K}{X^3} e^{X^2 + 2fX},$$

and so

$$e^{-x} v_X = \frac{K}{X^3} = \frac{K}{x}.$$

It follows that we can find a number M, such that

$$\left| e^{-x} u(x) \right| > \frac{K}{x}$$

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throughout the intervals

$$m^2,\ (m+\tfrac{1}{4})^2 \quad (m > M).$$

But
$$\Sigma \int_{m^2}^{(m+\frac{1}{4})^2} \frac{dx}{x} = 2\Sigma \int_{m}^{m+\frac{1}{4}} \frac{du}{u}$$

is evidently divergent. Hence

$$\int_0^\infty e^{-x} \mid u(x) \mid dx$$

is not convergent.

It is evident that we can apply the preceding analysis to a whole class of functions

$$u(x) = \sum_1^\infty \frac{(-)^i}{\phi(i)} \frac{x^{i^2}}{i^2!}$$

for which

$$\lim_{i=\infty} \frac{\phi(i+1)}{\phi(i)} = 1.$$

The behaviour of $u(x)$ for large values of x is dominated by that of its greatest term, and its modulus is generally comparable with

$$\frac{K}{\sqrt{x}\,\phi(\sqrt{x})}.$$

Taking, *e.g.*, $\phi(i) = i^\mu$ $(\mu > 0)$, the series

$$0 - 1 + 0 + 0 + \frac{1}{2^\mu} + 0 + 0 + 0 + 0 - \frac{1}{3^\mu} + \ldots$$

is not absolutely summable unless $\frac{1}{2}(\mu+1) > 1$ or $\mu > 1$, in which case the series is absolutely convergent.

It is easy to see that an absolutely convergent series must be absolutely summable; for

$$\mid u(x) \mid < \Sigma \frac{x^n}{n!} \mid u_n \mid,$$

and $\mid u_n \mid$ is convergent, so that

$$\int_0^\infty e^{-x} \mid u(x) \mid dx$$

is so. And as

$$u_\lambda + u_{\lambda+1} + \ldots$$

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is absolutely convergent, the same is true of

$$\int_0^\infty e^{-x} \left| \frac{d^\lambda}{dx^\lambda} u(x) \right| dx.$$

For a similar reason, if Σu_n is convergent, the integrals

$$\int_0^\infty e^{-x} \frac{d^\lambda}{dx^\lambda} u(x)\, dx$$

are convergent, even when

$$\int_0^\infty e^{-x} |u(x)|\, dx$$

is not.

Removal of terms from, and addition of terms to, a divergent series.

§ 4. M. Borel's alternative definition of the sum s of a divergent series $u_0 + u_1 + u_2 + \ldots$ is

$$s = \lim_{x=\infty} e^{-x} \sum_0^\infty \frac{s_n x^n}{n!}$$

where $$s_n = u_0 + u_1 + \ldots + u_n.$$

He proves that if the limit on the right is determinate

$$s - u_0 = \int_0^\infty e^{-x} \frac{d}{dx} u(x)\, dx$$

$$= u_1 + u_2 + \ldots,$$

according to the integral definition.

It does not follow that

$$s = u_0 + u_1 + u_2 + \ldots = \int_0^\infty e^{-x} u(x)\, dx.$$

We are thus led to consider the relation of the series

$$(1)\quad s = u_0 + u_1 + u_2 + \ldots;$$

$$(2)\quad s' = u_1 + u_2 + u_3 + \ldots,$$

and as M. Borel's discussion* of this question does not seem to me altogether satisfactory, I shall treat it in some detail.

* *Leçons*, pp. 100–103

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M. Borel's analysis is affected by the erroneous assumption that if

$$\int_0^\infty e^{-x} \mid u(x) \mid dx$$

is convergent, $e^{-x} u(x)$ has necessarily the limit 0 for $x = \infty$. And it seems to me that the notion of absolute summability, on which M. Borel lays considerable stress, does not, here at any rate, give us any real assistance.

I assume that $u(x)$ is an integral function. Then the sums of the series (1) and (2) are defined as

$$\int_0^\infty e^{-x} u(x)\, dx, \quad \int_0^\infty e^{-x} \frac{d}{dx} u(x)\, dx$$

respectively. Since

$$\int_0^X e^{-x} u(x)\, dx = -\left[e^{-x} u(x)\right]_0^X + \int_0^X e^{-x} u'(x)\, dx,$$

it follows that if

$$\lim_{x=\infty} e^{-x} u(x) = 0,$$

the summability of either (1) or (2) involves that of the other, and the relation

$$(3) \quad s = u_0 + s'.$$

Again, if both are summable, $e^{-x} u(x)$ has a limit for $x = \infty$, which can only be zero; so that (3) must be true.

But it can be shown that if (2) is summable, (1) must be so. The converse is not true; if, for instance

$$u_n = 2^n \sum_{\nu=0}^{\infty} \frac{(-)^\nu (\nu+1)^n}{2\nu+1\,!} = R\left[\frac{1}{i} \sum_{p=0}^{\infty} \frac{i^p (p+1)^n}{p!}\right],$$

$$u(x) = R\left[\frac{1}{i} \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{p=0}^{\infty} \frac{i^p (p+1)^n}{p!}\right]$$

$$= R\left[\frac{1}{i} \sum_{p=0}^{\infty} \frac{i^p}{p!} e^{(p+1)x}\right]$$

$$= e^x \sin e^x,$$

$$\sum_0^\infty u_n = \int_0^\infty \sin e^x\, dx$$

$$= \int_1^\infty \frac{\sin y}{y}\, dy,$$

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while $$\int_0^\infty e^{-x} \frac{d}{dx} \{e^x \sin e^x\}\, dx$$

is divergent.

I shall now prove that the summability of (1) follows from that of (2). In the first place I may remark that it is easy to give a valid form to M. Borel's proof of his proposition, that the absolute summability of (1) follows from that of (2). For, as M. Borel shows,* it is enough to prove this on the assumption that $u(x)$, $u'(x)$ are positive.

This being so, we can choose X so that

$$\int_X^{X'} e^{-x} u'(x)\, dx < \sigma,$$

however small be σ, for all values of $X' > X$. *A fortiori*

$$e^{-X'} \int_X^{X'} u'(x)\, dx < \sigma,$$

i.e. $$e^{-X'} \{u(X') - u(X)\} < \sigma.$$

But, X being fixed, we can choose $X_1 > X$ so that

$$e^{-X'} u(X) < \sigma$$

for all values of $X' > X_1$. Hence

$$e^{-X'} u(X') < 2\sigma \quad (X' > X_1).$$

Therefore

$$\lim_{x=\infty} e^{-x} u(x) = 0,$$

and the desired conclusion follows.

It is, however, almost equally easy to prove the more general theorem. For suppose that (2) is summable, and (1) not. Let

$$e^{-x} u(x) = \phi(x).$$

Then $$e^{-x} u'(x) = \phi(x) + \phi'(x).$$

Thus $$\int_0^\infty \{\phi(x) + \phi'(x)\}\, dx$$

is convergent, while $\int_0^\infty \phi(x)\, dx$ is not. Hence $\int_0^\infty \phi'(x)\, dx$

* *l.c.*, p. 101.

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is not convergent, so that $\phi(x)$ does not tend to any finite limit for $x=\infty$.

Now $\phi(x)$ either has or has not infinitely many zeroes in $(0, \infty)$. In the latter case it is ultimately of constant sign, say positive.

Then we can determine a positive quantity H, and an infinite series of intervals

$$x_\nu,\ x_\nu+\delta_\nu, \quad (x_{\nu+1} > x_\nu + \delta_\nu),$$

so that

$$(4) \quad \int_{x_\nu}^{x_\nu+\delta_\nu} \phi'(x)\,dx > H,$$

$(\nu = 1, 2, \ldots)$. This is obvious if

$$\overline{\lim_{x=\infty}}\ \phi(x) = \infty.$$

If this is not so, let the upper and lower limits of indetermination of $\phi(x)$ for $x=\infty$ be U, L. However small be the positive quantity σ, we can find two infinite series of quantities

$$\alpha_\nu, \beta_\nu\ (\alpha_{\nu+1} > \alpha_\nu, \beta_{\nu+1} > \beta_\nu, \lim_{\nu=\infty} \alpha_\nu = \infty, \lim_{\nu=\infty} \beta_\nu = \infty),$$

such that

$$\phi(\alpha_\nu) < L + \sigma,$$

$$\phi(\beta_\nu) > U - \sigma$$

$(\nu = 1, 2, \ldots)$. Hence we can find an infinite series of intervals $(x_\nu, x_\nu + \delta_\nu)$ as required, taking $H = U - L - 2\sigma$, and choosing the lower limits from the α's, and the upper limits from the β's.

A fortiori, as $\phi(x) > 0$,

$$\int_{x_\nu}^{x_\nu+\delta_\nu} \{\phi(x) + \phi'(x)\}\,dx > H.$$

This contradicts the hypothesis that

$$\int_0^\infty \{\phi(x) + \phi'(x)\}\,dx$$

is convergent.

There remains the possibility that $\phi(x)$ has infinitely many

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zeroes. One of its limits of indetermination must differ from 0; suppose that

$$\overline{\lim_{x=\infty}}\ \phi(x) = U > 0$$

(U may be $+\infty$). We can determine an infinite series of ascending quantities β_ν, whose limit is ∞, so that, if U is finite,

$$\phi(\beta_\nu) \geqq U - \sigma,$$

where σ is any small fixed positive quantity, or, if U is infinite,

$$\phi(\beta_\nu) \geqq G,$$

where G is any fixed positive quantity, however large. With each β_ν we associate the largest zero $< \beta_\nu$. Some zeroes will be associated with several of the points β; in this case we disregard all save the one which is nearest to the zero. We have thus an infinity of associated pairs of zeroes and points β; and it is evident that we can satisfy (4) by taking H to be $U - \sigma$ or G, x_ν to be one of the selected zeroes, and $x_\nu + \delta_\nu$ to be the associated point β. Moreover, $\phi(x)$ is positive throughout the intervals $(x_\nu, x_\nu + \delta_\nu)$; so that, as before,

$$\int_{x_\nu}^{x_\nu+\delta_\nu} \{\phi(x) + \phi'(x)\}\, dx > H.$$

It is therefore impossible that $\int_0^\infty \phi(x)\, dx$ should be divergent, and $\int_0^\infty \{\phi(x) + \phi'(x)\}\, dx$ convergent. Hence if (2) is summable, (1) is so too.

We conclude then that *if any one of the series*

$$\begin{aligned}
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&0 + 0 + 0 + u_0 + u_1 + \ldots,\\
&0 + 0 + u_0 + u_1 + u_2 + \ldots,\\
&0 + u_0 + u_1 + u_2 + u_3 + \ldots,\\
&u_0 + u_1 + u_2 + u_3 + u_4 + \ldots,\\
&u_1 + u_2 + u_3 + u_4 + u_5 + \ldots,\\
&u_2 + u_3 + u_4 + u_5 + u_6 + \ldots,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

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is summable, all those above it are summable, and their sums are related as if they were convergent.

I may remark, before leaving this part of the subject, that non-absolutely summable series are by no means 'artificial monstrosities.' Not only are the convergent series considered in § 4 of a natural and simple type, but so simple a divergent series as

$$\sum_0^\infty \frac{(1+i)^{n+1}}{n+1}$$

is non-absolutely summable,

Condition of consistency for other definitions.

§ 5. I shall now prove that the other definitions proposed by MM. Borel and Le Roy all satisfy the "condition of consistency.'

In the first place let us consider M. Borel's alternative definition mentioned at the beginning of the last paragraph. If $u(x)$ is an integral function

$$e^{-x}\sum_0^\infty \frac{s_n x^n}{n!} - u_0 = \int_0^x \frac{d}{dx}\left\{e^{-x}\sum_0^\infty \frac{s_n x^n}{n!}\right\} dx$$

$$= \int_0^x e^{-x}\sum_0^\infty \frac{u_{n+1}x^n}{n!}\, dx.$$

Hence, $\lim_{x=\infty} e^{-x}\Sigma\frac{s_n x^n}{n!}$ is finite and determinate if, and only if,

$$u_1 + u_2 + \ldots$$

is summable. If this is so

$$u_0 + u_1 + u_2 + \ldots$$

is summable. The latter series may be summable without $\lim. e^{-x}\Sigma\frac{s_n x^n}{n!}$ existing. But if it is convergent, $u_1 + u_2 + \ldots$ is convergent too, and

$$\lim_{x=\infty} e^{-x}\Sigma\frac{s_n x^n}{n!} = u_0 + u_1 + \ldots,$$

as may easily be proved directly.

M. Borel's generalised method of exponential summation

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presents no special difficulty. He defines the sum of the series $u_0 + u_1 + u_2 + \ldots$ as

$$\int_0^\infty e^{-x} u_k(x)\, dx,$$

$$u_k(x) = u_0 + u_1 + \ldots + u_{k-1} + (u_k + \ldots + u_{2k-1}) \frac{x}{1!} + \ldots.$$

According to this method, we associate the terms of the series in groups, and apply the ordinary exponential method to the series of groups; and as the terms of a convergent series may be grouped in any manner, the proof that the ordinary method satisfies the condition of consistency applies to this method also.*

M. le Roy suggests two generalisations of the exponential definition. According to one, we define $u_0 + u_1 + \ldots$ as

$$\frac{1}{p} \int_0^\infty e^{-x^{\frac{1}{p}}} x^{\frac{1}{p}-1} u_p(x)\, dx,$$

where $p > 0$, and

$$u_p(x) = \sum_0^\infty \frac{u_n x^n}{\Gamma(np+1)}.$$

This integral is equal to

$$\int_0^\infty e^{-y} \sum_0^\infty \frac{u_n y^{np}}{\Gamma(np+1)}\, dy,$$

if the latter is convergent. If p is an integer (the only case of much interest) the condition of consistency is certainly satisfied. For the integral is the sum of

$$u_0 + 0 + 0 + \ldots + u_1 + 0 + \ldots$$
$$+ u_2 + \ldots,$$

in which $p-1$ zero terms separate two consecutive u's, calculated by the ordinary method, and is therefore convergent and equal to the sum of this series, if

$$u_0 + u_1 + u_2 + \ldots$$

is convergent.

* The generalised exponential method is *not* always consistent with the ordinary method. If *e.g.* $k = 2$, it gives 0 as the sum of $1 - 1 + 1 - \ldots$. In fact we obtain $\frac{1}{2}$ or 0 according as k is odd or even.

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According to M. le Roy's other definition

$$u_0 + u_1 + \ldots = \lim_{t=1} \sum_0^\infty \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$$

$(0 < t < 1)$, if this limit is determinate. It may be expressed in the form

$$\lim_{t=1} \int_0^\infty e^{-x} \sum_0^\infty \frac{u_n x^{nt}}{n!}\, dx,$$

for, as I proved in § 3 of my former paper, integration term by term is in this case legitimate whenever the resulting series is convergent. Hence our series is defined as

$$\lim_{t=1} \int_0^\infty e^{-x} u(x^t)\, dx,$$

or

$$\lim_{t=1} \frac{1}{t} \int_0^\infty e^{-x^{\frac{1}{t}}} x^{\frac{1}{t}-1} u(x)\, dx,$$

or

$$(1) \quad \lim_{\alpha=0} \int_0^\infty e^{-x^{1+\alpha}} x^\alpha u(x)\, dx.$$

Now it can be shown that if

$$(2) \quad \int_0^\infty e^{-x} u(x)\, dx$$

is convergent, the integral in (1) is convergent for values of α in an interval $(0, \alpha_1)$, and that it tends to (2) for $\alpha = 0$.

For we can determine X so that

$$(3) \quad \left| \int_X^{X'} e^{-x} u(x)\, dx \right| < \sigma$$

$(X' > X)$. We can also suppose X so great* that $e^{x-x^{1+\alpha}} x^\alpha$ decreases steadily as x increases from X to ∞, for all values of α in a small interval $(0, \alpha')$. Then, by the second theorem of the mean

$$\int_X^{X_1} e^{-x^{1+\alpha}} x^\alpha u(x)\, dx$$

$$= e^{X - X^{1+\alpha}} X^\alpha \int_X^\xi e^{-x} u(x)\, dx$$

$$+ e^{X_1 - X_1^{1+\alpha}} X_1^\alpha \int_\xi^{X_1} e^{-x} u(x)\, dx$$

$(X \leqq \xi \leqq X_1)$, which is numerically less than 3σ.

* For this we need, in fact, only suppose $X \geqq 1$. Thus $e^{x-x^{1+\alpha}} x^\alpha \leqq 1$, $(x \geqq 1)$.

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Hence

$$(4) \quad \int_0^\infty e^{-x^{1+\alpha}} x^\alpha u(x)\, dx$$

is convergent. And it is equal to

$$\int_0^{X_2} + \int_{X_2}^\infty, \quad (X_2 > X).$$

The second part is equal to

$$e^{X_2 - X_2^{1+\alpha}} X_2^{\alpha} \int_{X_2}^{X_3} e^{-x} u(x)\, dx \quad (X_3 \geqq X_2),$$

which is numerically $< 2\sigma$. And we can choose α_1 so small that the first part differs from

$$\int_0^{X_2} e^{-x} u(x)\, dx$$

by less than σ, for all values of α in $(0, \alpha_1)$. Then, for such values of α, (4) differs from (2) by less than 5σ, so that

$$\lim_{\alpha=0} \int_0^\infty e^{-x^{1+\alpha}} x^\alpha dx = \int_0^\infty e^{-x} u(x)\, dx.$$

It follows that if $u_0 + u_1 + u_2 + \ldots$ is summable by the exponential method, it is summable also by M. le Roy's method, and that the two sums are the same. Hence M. le Roy's definition satisfies the condition of consistency.

Divergent Series and Mean Values.

§ 6. The 'mean value' of an infinite series of quantities $s_0, s_1, s_2, \ldots$ is

$$(1) \quad \lim_{n=\infty} \frac{s_0 + s_1 + \ldots + s_n}{n+1},$$

if this limit exists. It is well known that if $\lim. s_n$ exists and $= s$, the mean value also exists and $= s$. As the converse is not true, the expression (1) provides us with a generalisation of a notion of a limit or of the sum of a series from which interesting conclusions have been deduced.*

* Frobenius, *Crelle*, 89, p. 262; Hölder, *Math. Ann.*, 20, p. 535; Césaro, *Bull. des Sc. Math.*, 1890, p. 114.

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The question which I shall discuss in this paragraph is whether the existence of a mean value for

$$s_0, s_1, s_2, \ldots$$

involves the summability of the series

$$u_0 + u_1 + u_2 + \ldots.$$

If the mean value exists we may put

$$s_0 + s_1 + \ldots + s_n = (n+1)(s+\rho_n),$$
$$\lim_{n=\infty} \rho_n = 0.$$

Then

$$\sum_0^\infty \frac{s_n x^n}{n!} = \sum_0^\infty (s_0 + s_1 + \ldots + s_n)\left(\frac{x^n}{n!} - \frac{x^{n+1}}{n+1!}\right)$$
$$= \sum_0^\infty (n+1)(s+\rho_n)\left(\frac{x^n}{n!} - \frac{x^{n+1}}{n+1!}\right)$$
$$= se^x + \sum_0^\infty (n+1)\rho_n\left(\frac{x^n}{n!} - \frac{x^{n+1}}{n+1!}\right).$$

Since $\lim. \rho_n = 0$, we can choose a series of descending positive quantities ϵ_n such that

$$|\rho_{n'}| < \epsilon_n \quad (n' \geqq n),$$

and $\lim. \epsilon_n = 0$. Also if $\xi = [x]$

$$\sum_0^\infty (n+1)\rho_n\left(\frac{x^n}{n!} - \frac{x^{n+1}}{n+1!}\right)$$
$$= \sum_0^{X-1} + \sum_X^{\xi-1} + \sum_\xi^\infty \quad (1 < X < \xi).$$

All the coefficients of the quantities ρ_n in the first two sums are negative, except the last, which may be zero; and all in the third are positive. Hence

$$\left|\sum_0^{X-1} (n+1)\rho_n\left(\frac{x^n}{n!} - \frac{x^{n+1}}{n+1!}\right)\right|$$
$$< K\sum_0^{X-1} (n+1)\left(\frac{x^{n+1}}{n+1!} - \frac{x^n}{n!}\right)$$
$$< K\left\{\frac{Xx^X}{X!} - \left(1 + x + \ldots + \frac{x^{X-1}}{X-1!}\right)\right\}$$
$$< KX\frac{x^X}{X!},$$

where K is a constant.

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Again

$$\left| \sum_{X}^{\xi-1} (n+1)\,\rho_n \left(\frac{x^n}{n!} - \frac{x^{n+1}}{n+1!}\right) \right|$$

$$< \epsilon_X \sum_{X}^{\xi-1} (n+1) \left(\frac{x^{n+1}}{n+1!} - \frac{x^n}{n!}\right)$$

$$< \epsilon_X \left\{ \frac{\xi x^\xi}{\xi!} - \frac{x^{X+1}}{X+1!} - \ldots - \frac{x^{\xi-1}}{(\xi-1)!} - (X+1)\frac{x^X}{X!} \right\}$$

$$< \epsilon_X \frac{\xi x^\xi}{\xi!}.$$

Finally

$$\left| \sum_{\xi}^{\infty} (n+1)\,\rho_n \left(\frac{x^n}{n!} - \frac{x^{n+1}}{n+1!}\right) \right|$$

$$< \epsilon_\xi \sum_{\xi}^{\infty} (n+1) \left(\frac{x^n}{n!} - \frac{x^{n+1}}{n+1!}\right)$$

$$< \epsilon_\xi \left\{ (\xi+1)\frac{x^\xi}{\xi!} + \sum_{\xi}^{\infty} \frac{x^{n+1}}{n+1!} \right\}$$

$$< \epsilon_\xi \left\{ e^x + (\xi+1)\frac{x^\xi}{\xi!} \right\}.$$

Hence

$$(1) \quad \left| \sum_{0}^{\infty} (n+1)\,\rho_n \left(\frac{x^n}{n!} - \frac{x^{n+1}}{n+1!}\right) \right|$$

$$< KX\frac{x^X}{X!} + \epsilon_X \frac{\xi x^\xi}{\xi!} + \epsilon_\xi \left\{ e^x + (\xi+1)\frac{x^\xi}{\xi!} \right\}.$$

Now let $x = \xi + f$. Then when x is very large

$$\frac{x^\xi}{\xi!} = \frac{\xi^\xi \left(1+\frac{f}{\xi}\right)^\xi}{\xi^{\xi+\frac{1}{2}} e^{-\xi}\sqrt{2\pi}}(1+\alpha) = \frac{e^x}{\sqrt{(2\pi\xi)}}(1+\beta),$$

where α, β are very small. Hence the last term of (1) is equal to

$$\epsilon_\xi e^x \sqrt{\frac{\xi}{2\pi}}(1+\gamma),$$

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where γ is very small; and its product by e^{-x} will tend to zero for $x=\infty$, if

$$(2) \quad \lim_{\xi=\infty} e_\xi \sqrt{\xi} = 0;$$

and similarly the product of the second term by e^{-x} will vanish for $x=\infty$ if

$$(3) \quad \lim_{\xi=\infty} e_X \sqrt{\xi} = 0,$$

Finally, the same will be true of the first term if

$$(4) \quad \lim_{x=\infty} e^{-x} X \frac{x^X}{X!} = 0.$$

Suppose (2) satisfied, and $X=\frac{1}{2}\xi$, or $\frac{1}{2}(\xi+1)$, according as ξ is even or odd. Then, supposing ξ even, (3) will be satisfied, and as

$$\lim_{x=\infty} e^{(\xi+\gamma)} \left(\tfrac{1}{2}\xi\right) \frac{\xi^{\frac{1}{2}\xi}\left(1+\frac{\gamma}{\xi}\right)^{\frac{1}{2}\xi}}{\left(\frac{1}{2}\xi\right)^{\frac{1}{2}\xi+\frac{1}{2}} e^{-\frac{1}{2}\xi}\sqrt{2\pi}}$$

$$= \lim_{x=\infty} \tfrac{1}{2} e^{-\frac{1}{2}\gamma} \sqrt{\left(\frac{\xi}{\pi}\right)} \left(\frac{2}{e}\right)^{\frac{1}{2}\xi} = 0,$$

(4) is also satisfied.

Hence *if*

$$\frac{s_0 + s_1 + \ldots + s_n}{n+1} = s + \rho_n,$$

and

$$\lim_{n=\infty} \rho_n \sqrt{n} = 0,$$

then

$$\lim_{x=\infty} e^{-x} \sum_0^\infty \frac{s_n x^n}{n!} = s.$$

Thus the series $u_1 + u_2 + \ldots$, and *a fortiori* the series

$$u_0 + u_1 + u_2 + \ldots$$

is summable.

I may remark that the condition $\lim. \rho_n \sqrt{n} = 0$ is by no means *necessary*. In fact it is easy to construct *convergent* series for which it is not satisfied.

Suppose, *e.g.* $s_0 = 1$, and

$$s_n = 1 + \frac{1}{n^\epsilon} \quad (0 < \epsilon < \tfrac{1}{2}),$$

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Then $$\rho_n = \frac{\sum^{n} i^{-q}}{n+1},$$

and for $n = \infty$, ρ_n is of order

$$n^{-q},$$

so that $$\lim. \rho_n \sqrt{n} = \infty.$$

On the other hand the result would not be true if the condition were omitted. For we can choose the quantities s_n so that ρ_n is infinitely small of order $n^{-\frac{1}{2}}$ for $n = \infty$, and yet

$$\lim. e^{-x} \sum_{0}^{\infty} \frac{s_n x^n}{n!}$$

does not exist.

Suppose, *e.g.*

$$s_0 = 0,\ s_1 = -1,\ s_2 = s_3 = 0,\ s_4 = 2,$$

$$s_5 = \ldots = s_8 = 0,\ s_9 = -3,\ \ldots,$$

$$s_{n^2} = (-)^n n,\ \ldots.$$

Then we can prove as in § 3 that

$$e^{-x} \sum_{0}^{\infty} \frac{s_n x^n}{n!}$$

takes, as x increases to ∞, an infinite series of values, positive and negative, all numerically greater than a constant K. In fact, in the notation used near the end of § 3,

$$\frac{K}{\sqrt{x}\phi(\sqrt{x})} = K.$$

Now if $x = p^2 + q$ (p being the largest square contained in n)

$$s_0 + s_1 + \ldots + s_n = -1 + 2 - 3 + \ldots + (-)^p p$$

$$= \tfrac{1}{2}p \text{ or } -\tfrac{1}{2}(p+1),$$

according as p is even or odd.

Hence $$\left| \frac{s_0 + s_1 + \ldots + s_n}{n+1} \right| = \frac{\frac{1}{2}p}{p^2 + q + 1},$$

$$\text{or } \frac{\frac{1}{2}(p+1)}{p^2 + q + 1},$$

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i.e. is infinitely small for $n = \infty$ like

$$\frac{1}{p} \text{ or } \frac{1}{\sqrt{n}}.$$

The condition therefore corresponds to a real limitation, and has not been introduced by the method employed.

I shall not attempt now to prove any similar theorems concerning the more general mean values introduced by Hölder and Césaro. I will only remark that a similar line of argument may be applied to the more general question whether, if

$$\lim. \frac{s_0 + s_1 + \ldots + s_n}{n+1}$$

exists,

$$\lim_{x=\infty}. \frac{a_0 s_0 + a_1 s_1 x + a_2 s_2 x^2 + \ldots}{a_0 + a_1 x + a_2 x^2 + \ldots}$$

will exist and be equal to it,

$$a_0 + a_1 x + \ldots = \phi(x)$$

being an integral function.

For

$$\sum_0^\infty a_n s_n x^n = \sum_0^\infty (s_0 + s_1 + \ldots + s_n)(a_n x^n - a_{n+1} x^{n+1})$$

$$= \sum_0^\infty (n+1)(s + \rho_n)(a_n x^n - a_{n+1} x^{n+1})$$

$$= s\phi(x) + \sum_0^\infty (n+1)\rho_n (a_n x^n - a_{n+1} x^{n+1}).$$

Thus

$$\frac{\sum_0^\infty a_n s_n x^n}{\sum_0^\infty a_n x^n} = s + \frac{1}{\phi(x)} \sum_0^\infty (n+1)\rho_n (a_n x^n - a_{n+1} x^{n+1}).$$

We have then to prove that the limit of the last term for $x = \infty$ is zero, by a method analogous to that employed for the case in which $a_n = \frac{1}{n!}$, $\phi(x) = e^x$.

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The multiplication of divergent series.

§ 7. M. Borel has proved that if two series are absolutely summable, they may be multiplied according to the ordinary rule, and the validity of his proof is not affected by the considerations of §§ 3—4.

I showed in those paragraphs that in the treatment of certain questions fundamental in the theory we gained nothing by the introduction of the idea of absolute summability. This is, however, no longer true when we come to the question of the multiplication of series; I am, for instance, unable to prove that if two series Su_n, Sv_n are such that

$$u_\lambda + u_{\lambda+1} + \cdots,$$

$$v_\lambda + v_{\lambda+1} + \cdots$$

summable, whatever be λ, we may multiply them as if they were convergent; and indeed, if we consider the analogy of convergent series, it seems unlikely that this should be true.

There are, however, two interesting resemblances between the theories of the multiplication of convergent and divergent series. I may remark that in this connnection I consider *only* series which satisfy the condition mentioned above, for if a divergent series does not resemble a convergent one in this first and most elementary point, it is not of much interest to inquire whether it resembles one in other points more difficult to investigate.

This being understood, we can prove (1) that it is enough to suppose *one* series absolutely summable, and (2) that if the product series is summable, its sum is the product of the sums of the original series.

To prove (1) we have only to make a slight alteration in M. Borel's proof for the case in which both series are absolutely summable. The whole question turns* on whether we may assume that

$$(1)\quad \int_0^\infty e^{-x} u(x)\,dx \times \int_0^\infty e^{-y} v(y)\,dy = \int_0^\infty e^{-t}\left[\int_{-t}^{t} u\left(\frac{t+w}{2}\right) u\left(\frac{t-w}{2}\right)\frac{dw}{2}\right] dt.$$

Now the left-hand side is the limit for $H=\infty$ of

$$(2)\quad \int_0^H \int_0^H e^{-x-y} u(x)\, v(y)\,dx\,dy,$$

* *l.c.*, p. 104.

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and the right-hand side is the limit of

$$(2') \quad \iint e^{-x-y} u(x)\, v(y)\, dx\, dy$$

over the triangle bounded by

$$x = 0,\ y = 0, \text{ and } x + y = H.$$

Suppose that Su_n is absolutely summable. Then the difference of (2) and (2') is

$$\int_0^H e^{-x} u(x)\, dx \int_{H-x}^H e^{-y} v(y)\, dy.$$

Now
$$\left| \int_0^H e^{-x} u(x)\, dx \int_{H-x}^H e^{-y} v(y)\, dy \right|$$
$$\leqq \int_0^H \left\{ |e^{-x} u(x)| \left| \int_{H-x}^H e^{-y} v(y)\, dy \right| \right\} dx$$
$$\leqq \int_0^{\frac{1}{2}H} + \int_{\frac{1}{2}H}^H$$
$$\leqq \left| \int_{H-K}^H e^{-y} v(y)\, dy \right| \int_0^{\frac{1}{2}H} |e^{-x} u(x)|\, dx$$
$$+ \left| \int_{H-K'}^H e^{-y} v(y)\, dy \right| \int_{\frac{1}{2}H}^H |e^{-x} u(x)|\, dx,$$

where $$0 \leqq K \leqq \tfrac{1}{2}H, \quad \tfrac{1}{2}H \leqq K' \leqq H,$$

so that $\frac{1}{2}H \leqq H - K \leqq H$; and the limit of this for $H = \infty$ is 0.

The proof may then be completed as by M. Borel. It need only be observed that we do not prove that the product series is *absolutely* summable.

The second proposition (2) is most easily proved indirectly.* I shall first prove the following lemma.

If Su_n is summable, $Su_n t^n$ is absolutely summable for all positive values of t less than unity, and

$$\lim_{t=1} Su_n t^n = Su_n.$$

For† $$\lim_{x=0} e^{-x} \left(\frac{d}{dx}\right)^\lambda u(x) = 0 \quad (\lambda \geqq 0),$$

* The same is of course true of the corresponding theorem for convergent series.
† On account of the limitation made at the beginning of this paragraph.

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so that $$e^{-\frac{x}{t}}\left(\frac{d}{dx}\right)^{\lambda} u(x) \quad (0<t<1)$$

vanishes exponentially for $x=\infty$. Hence

$$\int_0^{\infty} e^{-x}\left(\frac{d}{dx}\right)^{\lambda} u(tx)\,dx = t^{\lambda-1}\int_0^{\infty} e^{-\frac{x}{t}}\left(\frac{d}{dx}\right)^{\lambda} u(x)\,dx$$

is absolutely convergent.

Also $$\lim_{t=1}\frac{1}{t}\int_0^{\infty} e^{-\frac{x}{t}} u(x)\,dx = \int_0^{\infty} e^{-x} u(x)\,dx$$

if the latter integral is convergent. This has been proved by Dirichlet.* The lemma follows.

I may remark that it is easy to prove in the same way a proposition analogous to Abel's well known theorem regarding series on their circle of convergence; viz. that *if a series* $Su_n x^n$ *is summable at any point* P *on the boundary of its polygon of summability, and* $f(P)$ *is its sum, the function* $f(x)$ *represented by the series within the polygon tends to* $f(P)$ *when* x *approaches* P *along the line* OP.

Now suppose that Su_n, Sv_n, and Sw_n or

$$u_0v_0 + (u_1v_0 + u_0v_1) + (u_2v_0 + u_1v_1 + u_0v_2) + \ldots$$

are all summable. Then Su_nt^n, Sv_nt^n, and Sw_nt^n are all absolutely summable if $0<t<1$, and

$$Su_nt_n \times Sv_nt^n = Sw_nt^n.$$

Hence in the limit

$$Su_n \times Sv_n = Sw_n.$$

§ 8. Before leaving this part of the subject I shall indicate another method by which the question of the multiplication of divergent series can be attacked. It is capable of more general application than is M. Borel's method, which depends upon a particular property of the exponential function, although, in the particular case of series whose sums are defined by the exponential method, it does not lead to quite such general conditions.

* v. Stolz, *Grundzüge*, I, p. 447; and my paper 'On the continuity and discontinuity of definite integrals,' etc. (*Q.J.*, 1902, p. 28).

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Suppose that the sum of the series

$$u_0 + u_1 + u_2 + \dots$$

is defined as

$$\int_0^\infty \phi(x) \sum_0^\infty a_n u_n x^n \, dx,$$

where

$$\int_0^\infty \phi(x) \, x^n \, dx = \frac{1}{a_n}.$$

Then the multiplication of the series Su_n, Sv_n may be justified by the following series of transformations:

$$Su_n \times Sv_n = Su_n \times \int_0^\infty \phi(y) \sum_0^\infty a_n v_n y^n \, dy$$

$$= \int_0^\infty \phi(y) \, dy \, [a_0 v_0 (u_0 + u_1 + u_2 + \dots)$$

$$+ a_1 v_1 y \ (0 + u_0 + u_1 + \dots)$$

$$+ a_2 v_2 y^2 (0 + 0 + u_0 + \dots)$$

$$+ \dots]$$

$$= \int_0^\infty \phi(y) \, dy \sum_0^\infty a_n v_n y^n \int_0^\infty \phi(x) \, dx \sum_0^\infty a_{n+m} u \ x^{n+m}$$

$$= \int_0^\infty \phi(y) \, dy \int_0^\infty \phi(x) \, dx \sum_0^\infty a_n r_n y^n \sum_0^\infty a_{n+m} u_m x^{n+m}$$

$$= \int_0^\infty \phi(x) \, dx \int_0^\infty \phi(y) \, dy \sum_0^\infty a_n v_n y^n \sum_0^\infty a_{n+m} u_m x^{n+m}$$

$$= \int_0^\infty \phi(x) \, dx \int_0^\infty \phi(y) \, dy \sum_0^\infty a_p x^p \sum_0^p a_i u_{p-i} v_i y^i$$

$$= \int_0^\infty \phi(x) \, dx \sum_0^\infty a_p x^p \int_0^\infty \phi(y) \, dy \sum_0^p a_i u_{p-i} v_i y^i$$

$$= \int_0^\infty \phi(x) \, dx \sum_0^\infty a_p w_p x^p$$

$$= Sw_p.$$

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Suppose, *e.g.*, that $a_n = \frac{1}{n!}$, $\phi(x) = e^{-x}$, and that we can find quantities H, ε, $(\varepsilon < 1)$, such that

$$|u(x)| < He^{\varepsilon x},$$
$$|v(y)| < He^{\varepsilon y}.$$

Then it is easy to see that

$$\left| \sum_0^\infty \frac{u_m x^{m+n}}{m+n!} \right| = \left| \left(\int_0^x dx \right)^n u(x) \right| < H\varepsilon^{-n} e^{\varepsilon x},$$

and hence that all the transformations are legitimate.

Part II.

DIVERGENT INTEGRALS.

Definitions of the value of a divergent integral.

§ 9. The various methods which I have considered in the preceding part of this paper enable us to attach a meaning to

$$\lim_{n=\infty} s_n, \quad u_0 + u_1 + \ldots + u_n + \ldots$$

in a number of cases in which they are otherwise meaningless. I shall now consider the possibility of framing similar conventions for the interpretation of the expressions

$$\lim_{x=\infty} \Phi(x), \quad \int^\infty \phi(x)\, dx.$$

Let us recall the process* which leads to M. Borel's definition of the sum of a divergent series. We take a doubly infinite sequence

$$a_n^{(p)},$$

such that $a_n^{(p)}$ increases with p for any fixed value of n, and decreases as n increases for any fixed value of p, and define the generalised limit of s_n as

$$\lim_{p=\infty} \frac{\sum_{n=0}^{\infty} a_n^{(p)} s_n}{\sum_{n=0}^{\infty} a_n^{(p)}}.$$

* Borel, *l.c.*, p. 95.

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In particular we take

$$a_n^{(p)} = \frac{p^n}{n!}.$$

The analogous method of defining lim. $\Phi(x)$ is evidently to define it as

$$(1) \quad \lim_{t=\infty} \frac{\int_0^\infty \theta(x, t)\, \Phi(x)\, dx}{\int_0^\infty \theta(x, t)\, dx},$$

$\theta(x, t)$ being a function of x and t which increases with t for for any fixed value of x (or at any rate for all values greater than a certain value), but decreases, when x increases and t is fixed, sufficiently rapidly to ensure the convergence of the integral in the numerator.

Further progress depends on our choice of a function $\theta(x, t)$.

§ 10. A natural assumption is

$$\theta(x, t) = p\left(\frac{x}{t}\right)^{p-1} e^{-\left(\frac{x}{t}\right)^p},$$

where p is any positive quantity. Then

$$\int_0^\infty \theta(x, t)\, dx = pt \int_0^\infty e^{-u^p} u^{p-1} du$$

$$= t,$$

so that (1) becomes

$$\lim_{t=\infty} \frac{p}{t} \int_0^\infty \left(\frac{x}{t}\right)^{p-1} e^{-\left(\frac{x}{t}\right)^p} \Phi(x)\, dx$$

$$= \lim_{t=\infty} \int_0^\infty e^{-u} \Phi(t\sqrt[p]{u})\, du.$$

From this we easily deduce a definition of a divergent integral

$$\int_0^\infty \phi(x)\, dx.$$

For if $\qquad \Phi(x) = \int_0^x \phi(x)\, dx,$

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we have defined $\lim_{n=\infty} \Phi(x)$ as

$$\lim_{t=\infty} \frac{p}{t} \int_0^\infty \left(\frac{x}{t}\right)^{p-1} e^{-\left(\frac{x}{t}\right)^p} \Phi(x)\, dx$$

$$= \lim_{t=\infty} \int_0^\infty e^{-\left(\frac{x}{t}\right)^p} \phi(x)\, dx,$$

provided $\lim_{x=\infty} \left(\frac{x}{t}\right)^{p-1} e^{-\left(\frac{x}{t}\right)^p} \Phi(x) = 0.$

The simplest case is that in which $p = 1$; and this assumption enables us to deal with many of the most interesting integrals which present themselves naturally in analysis. I shall therefore leave the general question on one side for the present, while I consider some consequences of the hypothesis

$$\theta(x, t) = e^{-\frac{x}{t}}.$$

I shall suppose that $\phi(x)$ is continuous for all finite values of x, and that

$$\lim_{x=\infty} e^{-\tau x} \phi(x) = 0$$

for any positive value of τ. And I define the *generalised limit* $L\Phi(x)$ by the equation

$$(1) \quad \underset{x=\infty}{L}\, \Phi(x) = \lim_{t=\infty} \int_0^\infty e^{-x} \Phi(tx)\, dx,$$

and the *divergent integral*

$$G \int_0^\infty \phi(x)\, dx$$

by the equation

$$(2) \quad G \int_0^\infty \phi(x)\, dx = \lim_{t=\infty} \int_0^\infty e^{-\frac{x}{t}} \phi(x)\, dx,$$

it being always understood that

$$\Phi(x) = \int^x \phi(x)\, dx.$$

Our hypotheses ensure that these definitions are consistent;

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and we may (as is easily seen) replace the right-hand side of (1) or (2) by the repeated integral

$$\int_0^\infty dt \int_0^\infty x e^{-tx} \phi(x)\, dx.$$

And it is evident that these definitions (and the more general definitions for which $p \neq 1$) are governed by the general principle of § 2.

Condition of consistency.

§ 11. It is easy to prove that if $\lim_{x=\infty}. \Phi(x)$ is determinate

$$L_{x=\infty} \Phi(x) = \lim_{x=\infty}. \Phi(x).$$

For
$$L\Phi(x) = \lim_{t=\infty}. \int_0^\infty e^{-x} \Phi(tx)\, dx$$

$$= \lim_{\tau=0}. \left(\tau \int_0^\infty e^{-\tau x} \Phi(x)\, dx\right)$$

$$= \lim_{\tau=0}. \tau \left(\int_0^X + \int_X^\infty\right),$$

if the last limit is determinate.

Now we can choose X so great that

$$|\Phi(x) - \Phi(\infty)| < \sigma$$

for all values of $x \geqq X$, however small be σ. Then

$$\tau \int_X^\infty e^{-\tau x} \Phi(x)\, dx = e^{-\tau X} \Phi(\infty) + \rho,$$

where
$$|\rho| < \sigma e^{-\tau X} < \sigma.$$

We can then choose τ_0 so that

$$\left|\tau \int_0^X\right| < \sigma$$

for all values of $\tau \leqq \tau_0$.

Thus
$$L\Phi(x) = \lim_{\tau=0}. e^{-\tau X} \phi(\infty)$$

$$= \phi(\infty).$$

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We may also prove directly that the condition of consistency is satisfied by our definition of

$$G\int_0^\infty \phi(x)\,dx$$

by using Dirichlet's theorem, mentioned in § 7, that

$$\lim_{\tau=0}\int_0^\infty e^{-\tau x}\phi(x)\,dx = \int_0^\infty \phi(x)\,dx,$$

whenever the last integral is convergent.

It is perhaps worth while at this point to state explicitly that $L\Phi(x)$ is *not* generally equal to

$$\lim_{x=\infty}\Phi(x_n),$$

$(x_0 < x_1 < x_2 < \ldots,\ \lim. x_n = \infty)$ when the latter limit is determinate, in the ordinary sense, or in M. Borel's sense. Nor is it generally true that

$$G\int_0^\infty \phi(x)\,dx = \sum_0^\infty \int_{x_n}^{x_{n+1}} \phi(x)\,dx$$

(supposing $x_0 = 0$).

Elementary properties of the generalised limit and the divergent integral.

§ 12. I. *If* $L\Phi_1(x)$, $L\Phi_2(x)$, ..., $L\Phi_n(x)$ *are determinate*

$$L\{a_1\Phi_1(x) + a_2\Phi_2(x) + \ldots + a_n\Phi_n(x)\}$$
$$= a_1 L\Phi_1(x) + a_2 L\Phi_2(x) + \ldots + a_n L\Phi_n(x).$$

This follows at once from the definition. It is obvious that a similar theorem holds for the divergent integral. On the other hand it is not generally true that

$$L\Phi(x)\,\Psi(x) = L\Phi(x)\,L\Psi(x).$$

when the two latter generalised limits exist.

II. *If* a *and* b *are constants, and* a *positive*

$$L\Phi(ax+b) = L\Phi(x)$$

if the latter is determinate.

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In the first place it follows immediately from the definition that

$$(1) \quad L\Phi(ax) = L\Phi(x).$$

Again
$$L\Phi(x+b) = \lim_{t=\infty} \int_0^\infty e^{-x} \Phi(tx+b)\, dx$$
$$= \lim_{t=\infty} e^{\frac{b}{t}} \int_{\frac{b}{t}}^\infty e^{-u} \Phi(tu)\, du.$$

But
$$\left| e^{\frac{b}{t}} \int_0^{\frac{b}{t}} e^{-u} \Phi(u)\, du \right| < K \int_0^{\frac{b}{t}} e^{-u}\, du,$$

and the limit of this for $t = \infty$ is plainly zero. Hence

$$(2) \quad L\Phi(x+b) = L\Phi(x).$$

From (1) and (2) the theorem follows.

It is easy to see that the corresponding theorem for divergent integrals is

$$G \int_0^\infty \phi(x)\, dx = a\, G \int_{-\frac{b}{a}}^\infty \phi(ax+b)\, dx,$$

$(a>0)$; *i.e. a divergent integral may be transformed by a linear substitution* $x = ay + b$ $(a>0)$ *according to the ordinary rule.*

Again from the equation

$$\int_0^\infty \{\phi(x)\psi'(x) + \phi'(x)\psi(x)\}\, dx = \phi(x)\psi(x) - \phi(0)\psi(0),$$

and Theorem I., it follows at once that

III. *A divergent integral may be transformed by integration by parts, if the process leads to a determinate result.*

For

$$G \int_0^\infty \{\phi(x)\psi'(x) + \phi'(x)\psi(x)\}\, dx = \underset{x=\infty}{L}\, \phi(x)\psi(x) - \phi(0)\psi(0),$$

if either of the two sides is determinate.

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IV. *If*

$$G\int_0^\infty \phi(x)\,dx$$

is determinate,

$$L\phi(x) = 0.$$

For $$L\phi(x) = \lim_{\tau=0}.\ \tau\int_0^\infty e^{-\tau x}\phi(x)\,dx = 0.$$

Generalised limits and mean values.

§ 13. The result of § 11 is capable of an extension naturally suggested by § 6.

We may define the *mean value of* $\Phi(x)$ *for* $x=\infty$ as

$$\lim_{x=\infty}.\frac{1}{x}\int_0^x \Phi(x)\,dx,$$

if this limit exists. In § 6 I proved that under certain conditions the fact that s_n possesses a mean value ensures that it possesses a generalised limit in M. Borel's sense. And it is easy to state conditions under which the fact that $\Phi(x)$ possesses a mean value ensures that it possesses a generalised limit in the sense of § 10.

In the first place, if $\Phi(x)$ possesses a mean value k, we can find constants X, K such that

$$\left|\int_0^x \Phi(x)\,dx\right| < Kx$$

for all values of $x \geqq X$. For if this were not so we could find two ascending sequences $X_1, X_2, \ldots, K_1, K_2, \ldots$ such that

$$\lim_{n=\infty}.\ X_n = \infty, \quad \lim_{n=\infty}.\ K_n = \infty,$$

and $$\left|\int_0^{X_n} \Phi(x)\,dx\right| > K_n X_n;$$

and then $\Phi(x)$ could not have a mean value.

Now $$\lim_{x=\infty}.\ e^{-\tau x}\Phi(x) = 0,$$

however small τ may be. Hence

$$\tau\int_0^\infty e^{-\tau x}\Phi(x)\,dx$$

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is convergent and equal to

$$\tau^2 \int_0^\infty \left(e^{-\tau x} \int_0^x \Phi(x)\, dx \right) dx.$$

Now we can choose $X_1 > X$ so that, if $x \geqq X_1$,

$$\int_0^x \Phi(x)\, dx = (k+\rho)\, x, \quad (|\rho| < \sigma),$$

however small be σ.

Then

$$\tau^2 \int_{X_1}^\infty e^{-\tau x} (k+\rho)\, x\, dx = k e^{-\tau X_1} (1 + \tau X_1) + R,$$

where $$|R| < \sigma e^{-\tau X_1} (1 + \tau X_1) < \sigma.$$

But

$$\left| \tau^2 \int_0^{X_1} \left(e^{-\tau x} \int_0^x \Phi(x)\, dx \right) dx \right| = \left| \tau^2 \left(\int_0^X + \int_X^{X_1} \right) \right|$$

$$< \tau^2 LX + \tau^2 K X_1^2,$$

where L is the maximum of $\left| e^{-\tau x} \int_0^x \Phi(x)\, dx \right|$ in $(0, X)$. Hence

$$\left| \tau \int_0^\infty e^{-\tau x} \Phi(x)\, dx - k \right|$$

$$< k \{1 - e^{-\tau X_1} (1 + \tau X_1)\} + \sigma + \tau^2 LX + \tau^2 K X_1^2,$$

and we can choose τ_0 so small that this is less than 2σ for all values of $\tau \leqq \tau_0$. Hence

$$\lim_{\tau = 0} \tau \int_0^\infty e^{-\tau x} \Phi(x)\, dx = k.$$

Thus, *e.g.* if $\Phi(x) = \cos x$, $k = 0$, and so

$$\lim_{x = \infty} \cos x = 0.$$

It is easy to prove that if $\Phi(\infty)$ is determinate

$$\lim_{x = \infty} \frac{1}{x} \int_0^x \Phi(x)\, dx = \Phi(\infty);$$

hence this result includes the result of § 11.

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If $\Phi(x)$ has no mean value it may still happen that

$$\frac{1}{x}\int_0^x \Phi(x)\,dx$$

has one; in this case we may define the mean value of $\Phi(x)$ as being equal to it; and so on generally.

If *e.g.* $$\Phi(x) = x\sin x,$$

$$\frac{1}{x}\int_0^x \Phi(x)\,dx = -\cos x + \frac{\sin x}{x},$$

and so the mean value of $\Phi(x)$ is 0. These extensions are analogous to those given by Hölder and Césaro to the notion of the mean value of s_n.

Some particular generalised limits and divergent integrals.

§ 14. I shall now find the value of

$$Le^{ax\imath} f(\sin^2 x),$$

$f(u)$ being any function of u continuous from $u=0$ to $u=1$, inclusive, and a being real.

By definition

$$Le^{ax\imath} f(\sin x) = \lim_{t=\infty} \int_0^\infty e^{-x(1-a\imath)} f(\sin tx)\,dx$$

$$= \lim_{\tau=0} \tau \int_0^\infty e^{-(\tau-a\imath)x} f(\sin^2 x)\,dx$$

$$= \lim_{\tau=0} \tau \sum_0^\infty \int_{n\pi}^{(n+1)\pi}$$

$$= \lim_{\tau=0} \tau \sum_0^\infty e^{-(\tau-a\imath)n\pi} \times \lim_{\tau=0} \int_0^\pi e^{-(\tau-a\imath)x} f(\sin^2 x)\,dx,$$

$$(1) \quad = \lim_{\tau=0} \frac{\tau}{1-e^{-(\tau-a\imath)\pi}} \int_0^\pi e^{ax\imath} f(\sin x)\,dx,$$

$$(2) \quad = \frac{1}{\pi}\int_0^\pi \cos ax\, f(\sin^2 x)\,dx,$$

or $$= 0,$$

according as a is or is not an even integer. Thus

$$L\sin ax\, f(\sin^2 x)$$

is always zero.

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Again

$$G\int_0^\infty e^{ax\sqrt{-1}} f(\sin^2 x)\,dx$$

$$= \lim_{\tau=0} \int_0^\infty e^{-(\tau-a\sqrt{-1})x} f(\sin^2 x)\,dx$$

$$= \lim_{\tau=0} \frac{1}{1-e^{-(\tau-a\sqrt{-1})\pi}} \int_0^\pi e^{-(\tau-a\sqrt{-1})x} f(\sin^2 x)\,dx.$$

If a is not an even integer this is

$$\frac{1}{1-e^{a\pi\sqrt{-1}}} \int_0^\pi e^{ax\sqrt{-1}} f(\sin^2 x)\,dx = \frac{i}{\sin \frac{1}{2}a\pi} \int_0^{\frac{1}{2}\pi} \cos ax\, f(\cos^2 x)\,dx.$$

Thus

$$(3) \quad G\int_0^\infty \cos ax\, f(\sin^2 x)\,dx = 0,$$

$$(4) \quad G\int_0^\infty \sin ax\, f(\sin^2 x)\,dx = \frac{1}{\sin \frac{1}{2}a\pi} \int_0^{\frac{1}{2}\pi} \cos ax\, f(\cos^2 x)\,dx.$$

On the other hand, if a is an even integer the first of these divergent integrals is not in general determinate, as

$$L \cos ax\, f(\sin^2 x)$$

is not in general zero. If, however,

$$\int_0^\pi \cos ax\, f(\sin^2 x)\,dx = 0,$$

we obtain

$$G\int_0^\infty \cos ax\, f(\sin^2 x)\,dx$$

$$= -\frac{1}{\pi} \int_0^\pi \cos ax\, f(\sin^2 x)\, x\,dx$$

$$= -\frac{1}{\pi} \int_0^\pi \cos ax\, f(\sin^2 x)\,(\pi - x)\,dx$$

$$= -\tfrac{1}{2} \int_0^\pi \cos ax\, f(\sin^2 x)\,dx$$

$$= 0.$$

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But

$$G\int_0^\infty \sin ax\, f(\sin^2 x)\, dx$$

$$= \lim_{\tau=0} \frac{i}{1-e^{-\tau\pi}} \int_0^\pi e^{-\tau x} \sin ax\, f(\sin^2 x)\, dx$$

$$(5) \qquad = -\frac{2}{\pi}(-)^{\frac{1}{2}a} \int_0^{\frac{1}{2}\pi} x \sin ax\, f(\cos^2 x)\, dx.$$

It may be pointed out, in connection with the remark at the end of § 11, that

$$G\int_0^\infty e^{axi} f(\sin^2 x)\, dx = \sum_0^\infty \int_{n\pi}^{(n+1)\pi} e^{axi} f(\sin^2 x)\, dx.$$

For the series is equal to

$$(1 + e^{a\pi i} + e^{2a\pi i} + \ldots) \int_0^\pi e^{axi} f(\sin^2 x)\, dx$$

$$= \frac{1}{1-e^{a\pi i}} \int_0^\pi e^{axi} f(\sin^2 x)\, dx,$$

provided a is not an even integer.

Some particular cases of the formulæ (1)—(5) are interesting. Thus

$$L\cos ax = L\sin ax = 0,$$

$$L(\cos^2 ax)^{\frac{1}{2}m} = L(\sin^2 ax)^{\frac{1}{2}m} = \frac{1}{\pi}\int_0^\pi (\cos^2 x)^{\frac{1}{2}m}\, dx$$

$$= \frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{m}{2}+1\right)},$$

if $m > 0$; and if $2n$ is a positive integer

$$L(\cos^2 x)^n = L(\sin^2 x)^n$$

$$= \frac{2}{\pi}\,\frac{2.4\ldots 2k}{3.5\ldots 2k+1}$$

or

$$= \frac{1.3\ldots 2k-1}{2.4\ldots 2k},$$

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according as $2n = 2k+1$ or $= 2k$. But

$$L(\cos x)^{2k+1} = L(\sin x)^{2k+1} = 0.$$

Some of these results may be easily deduced from first principles. Thus, *e.g.*, if $L\cos x$ is determinate, it must, by II., be equal to

$$L\cos(x+\pi) = -L\cos x,$$

and therefore $= 0$.

Again

$$G\int_0^\infty \cos ax\,dx = 0,$$

$$G\int_0^\infty \sin ax\,dx = \frac{1}{a},$$

$$G\int_0^\infty (\cos x)^{2k+1}\,dx = 0,$$

$$G\int_0^\infty (\sin x)^{2k+1}\,dx = \int_0^{\frac{1}{2}\pi} (\cos x)^{2k+1}\,dx$$

$$= \frac{2.4\ldots 2k}{3.5\ldots 2k+1}$$

Again

$$G\int_0^\infty \cos ax\,(\cos x)^{2k}\,dx = 0,$$

$$G\int_0^\infty \sin ax\,(\sin x)^{2k}\,dx$$

$$= \frac{1}{\sin\frac{1}{2}a\pi}\int_0^{\frac{1}{2}\pi} \cos au\,(\cos u)^{2k}\,du$$

$$= \frac{\pi}{2^{2k+1}\sin\frac{1}{2}a\pi}\,\frac{\Gamma(2k+1)}{\Gamma(k+1-\frac{1}{2}a)\,\Gamma(k+1+\frac{1}{2}a)}$$

$$= \frac{2k!}{a\,(2^2-a^2)\,(4^2-a^2)\ldots(4k^2-a^2)},$$

provided a is not an even integer.

Again

$$L\frac{1}{1-2p\cos x+p^2} = L\frac{1}{1-2p\cos 2x+p^2}$$

$$= \frac{1}{\pi}\int_0^\pi \frac{dx}{1-2p\cos 2x+p^2} = \frac{1}{1-p^2},$$

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$$or = \frac{1}{p^x - 1} \text{ (according as } p^2 < \text{or} > 1),$$

$$L \log (1 - 2p \cos x + p^2) = 0, \quad (p^2 < 1),$$
$$= 2 \log p, \quad (p^2 > 1),$$

$$G \int_0^\infty \log (1 - 2p \cos x + p^2)\, dx = 0, \quad (p^2 < 1),$$

and so on. Again, since

$$\int_0^\infty e^{-\tau x} x^{p-1} \begin{matrix}\cos \\ \sin\end{matrix} ax\, dx \quad (p > 1) = \frac{\Gamma(p)}{(\tau^2 + a^2)^{\frac{1}{2}p}} \begin{matrix}\cos \\ \sin\end{matrix} p \tan^{-1} \frac{a}{\tau},$$

$$G \int_0^\infty x^{p-1} \begin{matrix}\cos \\ \sin\end{matrix} ax\, dx = \frac{\Gamma(p)}{a^p} \begin{matrix}\cos \\ \sin\end{matrix} \tfrac{1}{2} p\pi.$$

If $p < 1$ these integrals are convergent in the ordinary sense. And $Lx^p \cos ax = Lx^p \sin ax = 0$, $(p > 0)$. But it is not necessary to multiply examples of the general formulæ.

A large number of these formulæ for divergent integrals were given by Raabe,* and are reproduced in Bierens de Haan's Tables. Raabe also gives 'limits' for $\cos x$, $\sin x$, etc., but not any of the general formulæ (1)—(5). He appears to think that he has proved, *e.g.*, that

$$\lim_{x=0} \cos x = 0$$

in the ordinary sense; his proof, which bears no resemblance to the methods of this paper, is of course wrong. It need hardly be said that at the time at which he wrote the idea of giving conventional definitions of such expressions as $\cos \infty$, $\sin \infty$, etc., had not occurred to anyone. It had of course been noticed that such expressions often presented themselves in analysis, and it had been found that there was generally one value which could be attributed to them without contradiction resulting; but there was as much confusion as to what was meant by such an equation as

$$\cos \infty = 0$$

as there was in the case of $1 - 1 + 1 - \ldots = \frac{1}{2}$. And the same is true of divergent integrals; in fact it would be very difficult to say in what sense Raabe regarded

$$\int_0^\infty \cos x\, dx$$

* *Integralrechnung*, I., pp. 280 *et seq.*

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as convergent. His results however, so far as they go, agree with those found here.

Raabe* discusses a more general integral

$$G\int_0^\infty e^{ax\iota} f(\sin^2\mu_1 x,\ \sin^2\mu_2 x,\ \ldots\ \sin^2\mu_n x)\,dx,$$

in which f is a continuous function, and $\mu_1, \mu_2, \ldots, \mu_n$ positive and rational. Raabe supposes $a=0$, and his analysis is invalid.

We may suppose that k is the L. C. M. of the denominators of $\mu_1, \ldots, \mu_n$. If $m_1 = k\mu_1$, $\mu_2 = k\mu_2, \ldots$, the integral is

$$kG\int_0^\infty e^{akx\iota} f(\sin^2 m_1 x,\ \ldots,\ \sin^2 m_n x)\,dx,$$

where now $m_1, \ldots, m_n$ are integers; this integral falls under the classes already treated. A similar reduction may be applied to

$$Le^{ax\iota} f(\sin^2\mu_1 x,\ \ldots,\ \sin^2\mu_n x).$$

§15. We have supposed so far that the functions $\Phi(x)$, $\phi(x)$ are continuous. But there is nothing to prevent us from attaching a sense to $L\Phi(x)$, even when $\Phi(x)$ becomes infinite for an infinity of indefinitely increasing values of x. We may agree that the definitions of $L\Phi(x)$ and

$$G\int_0^\infty \phi(x)\,dx$$

are to apply whenever the integrals which occur in them are convergent, or even if only their principal values are convergent. The formulæ worked out in §14 are, in general, the same under these extended conditions.

Thus, *e.g.*,

$$L\log(\cos x - \cos\alpha)^2 \quad (0<\alpha<\pi)$$

$$= \frac{1}{\pi}\int_0^\pi \log(\cos x - \cos\alpha)^2\,dx$$

$$= -2\log 2,$$

$$GP\int_0^\infty \frac{\sin x\,dx}{\cos x - \cos\alpha} = \log(4\sin^2\tfrac{1}{2}\alpha),$$

* *l.c.*, p. 310, and *Crelle*, XV., p. 355.

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a result which may be deduced either directly or from the fact that we must have

$$GP\int_0^\infty \frac{\sin x\,dx}{\cos x-\cos\alpha}=L\left[-\tfrac{1}{2}\log(\cos x-\cos\alpha)^2\right]_0^\pi.$$

Again $$L\frac{\cos ax}{\cos bx}=L\frac{\cos\frac{2a}{b}x}{\cos 2x}.$$

If $\frac{2a}{b}$ is not an even integer this is zero. If it is

$$L\frac{\cos\frac{2a}{b}x}{\cos 2x}=\frac{1}{\pi}P\int_0^\pi\frac{\cos\frac{2a}{b}x}{\cos 2x}dx$$

$$=0,\quad \left(\frac{a}{b}\text{ even}\right)$$

$$=(-1)^k,\quad \left(\frac{a}{b}=2k+1\right).$$

And $$GP\int_0^\infty\frac{\cos ax}{\cos bx}dx=\frac{2}{b}\,GP\int_0^\infty\frac{\cos\frac{2a}{b}x}{\cos 2x}\,dx=0$$

unless $\frac{a}{b}=2k+1$. In the case of

$$L\frac{\sin ax}{\sin bx}$$

we should have to modify our procedure, as

$$P\int_0^\pi\frac{\sin\frac{2a}{b}x}{\sin 2x}\,dx$$

is not generally convergent. But I shall not enter into this now, as the examples I have given will be sufficient to illustrate the different cases which may occur.

§ 16. The definitions of the previous sections are perhaps of most use in connection with double limit problems, such as differentiation under the integral sign. Their employment in

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such problems raises questions which demand a detailed treatment which I must reserve for the present. But they enable us to a considerable extent to disregard questions of convergence or divergence. I may, perhaps, illustrate this by showing how they can be used to calculate two important definite integrals.

(i) If
$$u = \int_0^\infty \frac{\cos ax}{1+x^2}\,dx, \quad (a > 0),$$
$$\frac{du}{da} = -\int_0^\infty \frac{x \sin ax}{1+x^2}\,dx,$$
$$\frac{d^2u}{da^2} = -G\int_0^\infty \frac{x^2 \cos ax}{1+x^2}\,dx,$$
$$\frac{d^2u}{da^2} - u = G\int_0^\infty \cos ax\,dx$$
$$= 0,$$
$$u = Ae^a + Be^{-a},$$

and as $u = \frac{1}{2}\pi$ for $a = 0$, and does not ever exceed $\frac{1}{2}\pi$,
$$u = \tfrac{1}{2}\pi e^{-a}.$$

(ii) If
$$u = P\int_0^\infty \frac{\cos ax}{\cos bx}\,\frac{dx}{1+x^2},$$

and
$$|a| \leqq |b|,$$
$$\frac{d^2u}{da^2} - u = -GP\int_0^\infty \frac{\cos ax}{\cos bx}\,dx = 0,$$
$$u = Ae^a + Be^{-a},$$

and as $u = \frac{1}{2}\pi$ for $a = \pm b$,
$$A = B = \frac{\pi}{4\cosh b},$$
$$u = \frac{\pi}{2}\,\frac{\cosh a}{\cosh b}.$$

This investigation of the value of (i) is that given in some of the older English text-books. The use of the definitions of this paper enables us to justify what appears to be a quite invalid line of argument.

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Other definitions.

§ 17. If we had taken

$$\theta(x, t) = p\left(\frac{x}{t}\right)^{p-1} e^{-\left(\frac{x}{t}\right)^p}, \quad (p \geqslant 1),$$

so that

$$L\Phi(x) = \lim_{t=\infty} \int_0^\infty e^{-u} \Phi(t \sqrt[p]{u})\, du,$$

and

$$G\int_0^\infty \phi(x)\, dx = \lim_{\tau=0} \int_0^\infty e^{-(\tau x)^p} \phi(x)\, dx,$$

we should have found the same formulæ as we deduced in § 15 on the hypothesis that $p = 1$, though tthe proofs would have been more difficult. But we should also have been able to attach meanings to limits and integrals which our actual definitions were not powerful enough to deal with.

Suppose for instance that $p = 2$, and

$$\phi(x) = e^{(a+bi)x} \quad (a > 0).$$

Then

$$\lim_{\tau=0} \int_{-\infty}^0 e^{-(\tau x)^2 + (a+bi)x}\, dx = \int_{-\infty}^0 e^{(a+bi)x}\, dx = \frac{1}{a+bi},$$

so that

$$G\int_0^\infty \phi(x)\, dx = \lim_{\tau=0} \int_{-\infty}^\infty e^{-(\tau x)^2 + (a+bi)x}\, dx - \frac{1}{a+bi}.$$

But

$$\int_{-\infty}^\infty e^{-(\tau x)^2 + (a+bi)x}\, dx = e^{\left(\frac{a}{2\tau}\right)^2} \int_{-\infty}^\infty e^{-(\tau x)^2 + bxi}\, dx$$

$$= \frac{\sqrt{\pi}}{\tau} e^{\left(\frac{a}{2\tau}\right)^2 - \left(\frac{b}{2\tau}\right)^2}.$$

If $a^2 < b^2$, the limit of this for $\tau = 0$ is 0, and so

$$G\int_0^\infty e^{(a+bi)x}\, dx = -\frac{1}{a+bi},$$

and

$$Le^{(a+bi)x} = 0.$$

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It is of course clear that if we are to give any definition of $L\Phi(x)$ which is consistent with itself, it must give

$$\begin{aligned} Le^{(a+bi)x} &= Le^{(a+bi)(x+\pi)} \\ &= e^{(a+bi)\pi} Le^{(a+bi)x} \end{aligned}$$

and so $$= 0.$$

And it is very desirable to find a convention which will give

$$\underset{x=\infty}{Le^{zx}} = 0,$$

$$G\int_0^\infty e^{zx}dz = -\frac{1}{z}$$

for as large a region of values of $z = a + bi$ as possible. This region is evidently that formed by all values of z other than real and positive values. For it is only when z is real and positive that

$$\int_0^\infty e^{zx}dx$$

is definitely infinite.

§ 18. Now M. le Roy has proved, in the memoir already cited, that if $t < 1$

$$\lim_{t=1}\int_0^\infty e^{-x+tzx}dx = \frac{1}{1-z}$$

for all values of z except real values > 1. And by a similar method we can prove that

$$\lim_{\tau=0}\int_0^\infty e^{-\tau x\log x+zx}dx = -\frac{1}{z}$$

for all values of z except real and positive values.

In the first place, it is clear that

$$\lim_{\tau=0}\int_0^\infty e^{-\tau x\log x+zx}dx = -\frac{1}{z},$$

if $R(z) < 0$. But

$$\int_0^\infty e^{-\tau x\log x+zx}dx = \int_0^\infty \exp.\{-\tau\rho e^{i\phi}\log(\rho e^{i\phi}) + z\rho e^{i\phi} + i\phi\}d\rho,$$

if $0 < \phi < \frac{1}{2}\pi$, and the logarithm has its principal value. To prove this we integrate

$$\int e^{-\tau x\log x+zx}dx \quad (x = \xi + i\eta = \rho e^{i\theta})$$

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round the contour bounded by the lines

$$\theta = 0,\ \theta = \phi,\ \rho = \epsilon,\ \rho = R,$$

ϵ being very small and R very large. All that is necessary is to prove that the integral over the arc $\rho = R$ tends to zero for $R = \infty$. This follows from the fact that when ρ is large the subject of integration is of order

$$\text{exp. } \rho \{-\tau \log \rho \cos\theta + \tau\theta \sin\theta + a\cos\theta - b\sin\theta\}.$$

Now

$$\lim_{\tau=0} \int_0^\infty \text{exp. } \{-\tau\rho e^{i\phi} \log(\rho e^{i\phi}) + z\rho e^{i\phi} + i\phi\}\, d\rho$$

$$= \int_0^\infty \text{exp. } \{z\rho e^{i\phi} + i\phi\}\, d\rho$$

$$= -\frac{1}{z},$$

provided the second integral is convergent. This will be the case if $a\cos\phi - b\sin\phi < 0$. The line $\xi\cos\phi = \eta\sin\phi$ is the line through 0, making an angle $\frac{1}{2}(\pi - \phi)$ with the real axis; and $a\cos\phi - b\sin\phi < 0$ if z lies to the left of this line. But ϕ may have any value between 0 and $\frac{1}{2}\pi$, and as ϕ varies between these limits the region to the left of the different positions of the line covers all the plane except the fourth quadrant. Thus (1) holds except when z lies in the fourth quadrant. By supposing ϕ negative we can prove that it holds except in the first quadrant. It therefore holds except when z is on the real axis.

Thus if we define

$$G\int_0^\infty \phi(x)\, dx,$$

as

$$\lim_{\tau=0} \int_0^\infty e^{-\tau x \log x}\, \phi(x)\, dx,$$

we find that

$$G\int_0^\infty e^{(a+bi)x}\, dx = -\frac{1}{a+bi}$$

except when $a > 0$, $b = 0$.

If we start from the equation

$$\lim_{t=0} \int_0^\infty e^{-x+(1+z)x^t}\, dx = -\frac{1}{z},$$

and put

$$x^t = u,\ \frac{1}{t} = 1 + \tau,$$

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we obtain

$$\lim_{\tau=0} \int_0^\infty u^\tau e^{u-u^{1+\tau}+uz} du = -\frac{1}{z},$$

or

$$\lim_{\tau=0} \int_0^\infty (1+\ldots)\, e^{-\tau u \log u - \ldots} e^{uz} du = -\frac{1}{z}.$$

This shows the connection between the equation proved by M. le Roy and that proved here.

Suppose that $\Phi(0)=0$. Then the corresponding definition of $L\Phi(x)$ is

$$L\Phi(x) = \lim_{\tau=0} \int_0^\infty e^{-\tau x \log x} \Phi'(x)\, dx$$

$$= \lim_{\tau=0} \int_0^\infty \tau (\log x + 1)\, e^{-\tau x \log x} \Phi(x)\, dx,$$

if $\lim_{x=\infty} e^{-\tau x \log x} \Phi(x) \log x = 0$.

If we observe that

$$\int_0^\infty (\log x + 1)\, e^{-\tau x \log x}\, dx = \frac{1}{\tau},$$

we see that this definition agrees with that given by the general formula of § 9, if

$$\theta(x, t) = (\log x + 1)\, e^{-\frac{x}{t} \log x}$$

$$= t \frac{d}{dx} \left(e^{-\frac{x}{t} \log x} \right).$$

Much more general results are given by this choice of $\theta(x, t)$ than by the more simple one adopted in §§ 11–16. But the latter enables us to deal with the simplest and most obvious cases with greater facility.

At this point I shall, for the present, bring these investigations to a close. My object in the second part of this paper has been rather to discuss the general principles which must govern our conventions, and to illustrate the different range of different definitions, than to enter into great detail concerning any one of them. The subject is a large one, and I hope to return to it on some future occasion.

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SUGGESTIONS TOWARDS THE FORMATION OF A GENERAL THEORY OF SYSTEMS OF PFAFFIAN EQUATIONS.

By J. BRILL, M.A.

PART V.*

Further Development of Conditioned Cases.

§ 30. THE method adopted in article 22 is capable of extension to more complicated cases, and I propose in the present Part to follow out this line of investigation somewhat further. In Part II. I gave a full discussion of the case in which the number of integrals is the same as the number of equations. For the case in which the number of integrals is one more than the number of equations, the investigation will follow closely the lines of article 22, but we shall have a more complicated analysis when the number of integrals is greater than this.

As the A notation of the early Parts is only convenient for special cases, I will adopt the notation introduced in Part IV., replacing the round brackets by double ones, as the latter will admit of the use of round brackets within them, which is sometimes of convenience.

Thus we will write

$$\{123\ldots(n+1)\} = \frac{\partial}{\partial x_1}(2, 3, 4, \ldots, n+1) - \frac{\partial}{\partial x_2}(1, 3, 4, \ldots, n+1)$$

$$+\ldots+(-1)^n \frac{\partial}{\partial x_{n+1}}(1, 2, 3, \ldots, n),$$

and

$$\{r, 123\ldots(n+2)\} = a_{r1}\{234\ldots(n+2)\} - a_{r2}\{134\ldots(n+2)\}$$

$$+\ldots+(-1)^{n+1} a_{r(n+2)}\{123\ldots(n+1)\}.$$

Suppose that we have n equations in m variables, and that

* Part I. appeared in Vol. XXX., p. 221; Part II. in Vol. XXXII., p. 183; Part III. in Vol. XXXIV., p. 53; and Part IV. in Vol. XXXIV., p. 155.

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the number of functions in the integral system is p. To simplify notation we will write, where convenient,

$$p = n + h,\ m = n + k,\ k = h + q.$$

Thus our original set of equations will be equivalent to the set

$$\begin{aligned}
&(1, k+2, k+3, \ldots, m)\,dx_1 + (2, k+2, k+3, \ldots, m)\,dx_2 \\
&\qquad + \ldots + (k, k+2, k+3, \ldots, m)\,dx_k \\
&\qquad + (k+1, k+2, \ldots\ldots, m)\,dx_{k+1} = 0, \\
&(1, k+1, k+3, k+4, \ldots, m)\,dx_1 + (2, k+1, k+3, k+4, \ldots, m)\,dx_2 \\
&\qquad + \ldots + (k, k+1, k+3, k+4, \ldots, m)\,dx_k \\
&\qquad - (k+1, k+2, \ldots\ldots, m)\,dx_{k+2} = 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
&(1, k+1, k+2, \ldots, r-1, r+1, \ldots, m)\,dx_1 \\
&\qquad + (2, k+1, k+2, \ldots, r-1, r+1, \ldots, m)\,dx_2 \\
&\qquad + \ldots + (k, k+1, \ldots, r-1, r+1, \ldots, m)\,dx_k \\
&\qquad + (-1)^{r-k-1}(k+1, k+2, \ldots\ldots\ldots, m)\,dx_r = 0, \\
&\qquad\qquad \&c.
\end{aligned}$$

Now the number of equations in the integral system being p, we may express p of the variables in terms of the remaining $m - p$, or q. We will denote the differential coefficients taken on this supposition by symbols of the type

$$\frac{dx_r}{dx_f},$$

where x_f is one of the selected set of q variables, and x_r is one of the remaining p variables. Further we will suppose the selected set to be constituted of the first q variables.

Thus the typical equation of the above set gives rise to a set of q equations of the type

$$\begin{aligned}
&(f, k+1, k+2, \ldots, r-1, r+1, \ldots, m) \\
&\qquad + (q+1, k+1, k+2, \ldots, r-1, r+1, \ldots, m)\frac{dx_{q+1}}{dx_f} \\
&\qquad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\qquad + (k, k+1, k+2, \ldots, r-1, r+1, \ldots, m)\frac{dx_k}{dx_f} \\
&\qquad + (-1)^{r-k-1}(k+1, k+2, \ldots\ldots\ldots, m)\frac{dx_r}{dx_f} = 0 \quad \ldots(59).
\end{aligned}$$

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From these equations we can deduce another by means of the relations of the type

$$\frac{d^2x_r}{dx_f dx_g} = \frac{d^2x_r}{dx_g dx_f} \quad \ldots\ldots\ldots\ldots\ldots\ldots (60).$$

To simplify the notation as much as possible we will choose, for working out in full, the equation deduced from the relation

$$\frac{d^2x_{k+1}}{dx_1 dx_2} = \frac{d^2x_{k+1}}{ax_2 dx_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (61).$$

We have

$$-\frac{dx_{k+1}}{dx_1} = \frac{(1, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)} + \frac{(q+1, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)} \frac{dx_{q+1}}{dx_1}$$
$$+\ldots+ \frac{(k, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)} \frac{dx_k}{dx_1},$$

and

$$-\frac{dx_{k+1}}{dx_2} = \frac{(2, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)} + \frac{(q+1, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)} \frac{dx_{q+1}}{dx_2}$$
$$+\ldots+ \frac{(k, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)} \frac{dx_k}{dx_2}.$$

Combining these two results with the aid of equation (61), and taking account of the other equations of the type (60), we obtain

$$\left\{\frac{\partial}{\partial x_2} + \frac{dx_{q+1}}{dx_2}\frac{\partial}{\partial x_{q+1}} + \ldots + \frac{dx_k}{dx_2}\frac{\partial}{\partial x_k} + \Sigma \frac{dx_r}{dx_2}\frac{\partial}{\partial x_r}\right\} \frac{(1, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)}$$
$$+ \frac{dx_{q+1}}{dx_1}\left\{\frac{\partial}{\partial x_2} + \frac{dx_{q+1}}{dx_2}\frac{\partial}{\partial x_{q+1}} + \ldots + \frac{dx_k}{dx_2}\frac{\partial}{\partial x_k} + \Sigma \frac{dx_r}{dx_2}\frac{\partial}{\partial x_r}\right\}$$
$$\times \frac{(q+1, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \frac{dx_k}{dx_1}\left\{\frac{\partial}{\partial x_2} + \frac{dx_{q+1}}{dx_2}\frac{\partial}{\partial x_{q+1}} + \ldots + \frac{dx_k}{dx_2}\frac{\partial}{\partial x_k} + \Sigma \frac{dx_r}{dx_2}\frac{\partial}{\partial x_r}\right\}$$
$$\times \frac{(k, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)}$$

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$$= \left\{\frac{\partial}{\partial x_1} + \frac{dx_{q+1}}{dx_1}\frac{\partial}{\partial x_{q+1}} + \ldots + \frac{dx_k}{dx_1}\frac{\partial}{\partial x_k} + \Sigma\frac{dx_r}{dx_1}\frac{\partial}{\partial x_r}\right\}\frac{(2, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)}$$

$$+ \frac{dx_{q+1}}{dx_2}\left\{\frac{\partial}{\partial x_1} + \frac{dx_{q+1}}{dx_1}\frac{\partial}{\partial x_{q+1}} + \ldots + \frac{dx_k}{dx_1}\frac{\partial}{\partial x_k} + \Sigma\frac{dx_r}{dx_1}\frac{\partial}{\partial x_r}\right\}$$

$$\times \frac{(q+1, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{dx_k}{dx_2}\left\{\frac{\partial}{\partial x_1} + \frac{dx_{q+1}}{dx_1}\frac{\partial}{\partial x_{q+1}} + \ldots + \frac{dx_k}{dx_1}\frac{\partial}{\partial x_k} + \Sigma\frac{dx_r}{dx_1}\frac{\partial}{\partial x_r}\right\}$$

$$\times \frac{(k, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)}$$

$$\ldots\ldots\ldots(62).$$

In this equation we have to substitute, for the quantities of the type

$$\frac{dx_r}{dx_1} \text{ and } \frac{dx_r}{dx_2},$$

their values as obtained from the equations of the type (59). On reduction it will be seen that all the terms of the type

$$\frac{dx_u}{dx_1}\frac{dx_u}{dx_2},$$

where u is one of the numbers

$$q+1,\ q+2,\ \ldots,\ k,$$

disappear. Further, if v be another of this set of numbers, we have a set of terms of the type

$$\frac{d(x_u, x_v)}{d(x_1, x_2)}\left[\frac{\partial}{\partial x_v}\frac{(u, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)} - \frac{\partial}{\partial x_u}\frac{(v, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)}\right.$$

$$+ \Sigma(-1)^{r-k}\left\{\frac{(v, k+1, k+2, \ldots, r-1, r+1, \ldots, m)}{(k+1, k+2, \ldots, m)}\right.$$

$$\times \frac{\partial}{\partial x_r}\frac{(u, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)}$$

$$\left.\left.- \frac{(u, k+1, k+2, \ldots, r-1, r+1, \ldots, m)}{(k+1, k+2, \ldots, m)}\frac{\partial}{\partial x_r}\frac{(v, k+2, k+3, \ldots, m)}{(k+1, k+2, \ldots, m)}\right\}\right].$$

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Considering those terms in the coefficient of

$$\frac{d(x_u, x_v)}{d(x_1, x_2)},$$

that involve the symbol

$$\frac{\partial}{\partial x_r},$$

we see that they may be written in the form

$$\frac{(-1)^{r-k} U}{\{(k+1, k+2, \ldots, m)\}^2},$$

where U stands for the expression

$$\left\{(k+1, k+2, \ldots, m)\frac{\partial}{\partial x_r} - \frac{\partial(k+1, k+2, \ldots, m)}{\partial x_r}\right\}$$

$$\times \Big\{(v, k+1, k+2, \ldots, r-1, r+1, \ldots, m)(u, k+2, k+3, \ldots, m)$$

$$-(u, k+1, k+2, \ldots, r-1, r+1, \ldots, m)(v, k+2, k+3, \ldots, m)$$

$$+(k+1, k+2, \ldots, m)(u, v, k+2, k+3, \ldots, r-1, r+1, \ldots, m)\Big\}$$

$$-(k+1, k+2, \ldots, m)$$

$$\times \Big\{(u, k+2, k+3, \ldots, m)\frac{\partial}{\partial x_r}(v, k+1, k+2 \ldots, r-1, r+1, \ldots, m)$$

$$-(v, k+2, k+3, \ldots, m)\frac{\partial}{\partial x_r}(u, k+1, k+2, \ldots, r-1, r+1, \ldots, m)$$

$$+(k+1, k+2, \ldots, m)\frac{\partial}{\partial x_r}(u, v, k+2, k+3, \ldots, r-1, r+1, \ldots, m)\Big\}.$$

Thus, in virtue of the relation

$$(v, k+1, k+2, \ldots, r-1, r+1, \ldots, m)(u, k+2, k+3, \ldots, m)$$

$$-(u, k+1, k+2, \ldots, r-1, r+1, \ldots, m)(v, k+2, k+3, \ldots, m)$$

$$+(k+1, k+2, \ldots, m)(u, v, k+2, k+3, \ldots, r-1, r+1, \ldots, m) = 0,$$

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we find that the coefficient of

$$\frac{d(x_u, x_v)}{d(x_1, x_2)}$$

reduces to

$$\frac{-1}{\{(k+1, k+2, \ldots, m)\}^2}[(k+1, k+2, \ldots, m)\{uv(k+2)(k+3)\ldots m\}$$
$$-(v, k+2, k+3, \ldots, m)\{u(k+1)(k+2)\ldots m\}$$
$$+(u, k+2, k+3, \ldots, m)\{v(k+1)(k+2)\ldots m\}].$$

In a similar manner we obtain, for the coefficient of

$$\frac{dx_u}{dx_2},$$

the expression

$$\frac{-1}{\{(k+1, k+2, \ldots, m)\}^2}[(k+1, k+2, \ldots, m)\{1u(k+2)(k+3)\ldots m\}$$
$$-(u, k+2, k+3, \ldots, m)\{1(k+1)(k+2)\ldots m\}$$
$$+(1, k+2, k+3, \ldots, m)\{u(k+1)(k+2)\ldots m\}],$$

and for that of

$$\frac{dx_u}{dx_1},$$

the expression

$$\frac{1}{\{(k+1, k+2, \ldots, m)\}^2}[(k+1, k+2, \ldots, m)\{2u(k+2)(k+3)\ldots m\}$$
$$-(u, k+2, k+3, \ldots, m)\{2(k+1)(k+2)\ldots m\}$$
$$+(2, k+2, k+3, \ldots, m)\{u(k+1)(k+2)\ldots m\}].$$

Finally those terms of our equation, that do not contain any differential coefficients of the type

$$\frac{dx_u}{dx_f},$$

reduce to

$$\frac{-1}{\{(k+1, k+2, \ldots, m)\}^2}[(k+1, k+2, \ldots, m)\{12(k+2)(k+3)\ldots m\}$$
$$-(2, k+2, k+3, \ldots, m)\{1(k+1)(k+2)\ldots m\}$$
$$+(1, k+2, k+3, \ldots, m)\{2(k+1)(k+2)\ldots m\}].$$

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Thus, if we assume that the determinant

$$(k+1,\ k+2,\ \ldots,\ m)$$

is finite, equation (62) becomes

$$\begin{aligned}
&\Sigma \frac{d(x_u, x_v)}{d(x_1, x_2)} [(k+1, k+2, \ldots, m)\ \{uv\,(k+2)\,(k+3)\ldots m\} \\
&\qquad -(v,\ k+2,\ k+3,\ \ldots,\ m)\ \{u\,(k+1)\,(k+2)\ldots m\} \\
&\qquad +(u,\ k+2,\ k+3,\ \ldots,\ m)\ \{v\,(k+1)\,(k+2)\ldots m\}] \\
&-\Sigma \frac{dx_u}{dx_1} [(k+1, k+2, \ldots, m)\ \{2u\,(k+2)\,(k+3)\ldots m\} \\
&\qquad -(u,\ k+2,\ k+3,\ \ldots,\ m)\ \{2\,(k+1)\,(k+2)\ldots m\} \\
&\qquad +(2,\ k+2,\ k+3,\ \ldots,\ m)\ \{u\,(k+1)\,(k+2)\ldots m\}] \\
&+\Sigma \frac{dx_u}{dx_2} [(k+1, k+2, \ldots, m)\ \{1u\,(k+2)\,(k+3)\ldots m\} \\
&\qquad -(u,\ k+2,\ k+3,\ \ldots,\ m)\ \{1\,(k+1)\,(k+2)\ldots m\} \\
&\qquad +(1,\ k+2,\ k+3,\ \ldots,\ m)\ \{u\,(k+1)\,(k+2)\ldots m\}] \\
&+(k+1,\ k+2,\ \ldots,\ m)\ \{12\,(k+2)\,(k+3)\ldots m\} \\
&\qquad -(2,\ k+2,\ k+3,\ \ldots,\ m)\ \{1\,(k+1)\,(k+2)\ldots m\} \\
&\qquad +(1,\ k+2,\ k+3,\ \ldots,\ m)\ \{2\,(k+1)\,(k+2)\ldots m\} \\
&\qquad\qquad = 0 \ldots\ldots\ldots (63).
\end{aligned}$$

§ 31. We now proceed to put equation (63) into a different form. For this purpose we shall require certain identities which we proceed to establish.

We have a set of equations of the form

$$\begin{aligned}
&a_{21}(2, 3, 4, \ldots, n+1) - a_{22}(1, 3, 4, \ldots, n+1) \\
&\qquad +\ldots+ (-1)^n a_{2(n+1)}(1, 2, 3, \ldots, n) = 0, \\
&a_{31}(2, 3, 4, \ldots, n+1) - a_{32}(1, 3, 4, \ldots, n+1) \\
&\qquad +\ldots+ (-1)^n a_{3(n+1)}(1, 2, 3, \ldots, n) = 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&a_{n1}(2, 3, 4, \ldots, n+1) - a_{n2}(1, 3, 4, \ldots, n+1) \\
&\qquad +\ldots+ (-1)^n a_{n(n+1)}(1, 2, 3, \ldots, n) = 0.
\end{aligned}$$

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From these we can eliminate all the determinants of the type $(2, 3, 4, \ldots, n+1)$ except three. Thus, choosing the first three as those to remain, and utilizing the notation of Part I., we obtain

$$\begin{pmatrix} 1, & 4, & 5, & \ldots, & n+1 \\ 2, & 3, & 4, & \ldots, & n \end{pmatrix} (2, 3, 4, \ldots, n+1)$$
$$- \begin{pmatrix} 2, & 4, & 5, & \ldots, & n+1 \\ 2, & 3, & 4, & \ldots, & n \end{pmatrix} (1, 3, 4, \ldots, n+1)$$
$$+ \begin{pmatrix} 3, & 4, & 5, & \ldots, & n+1 \\ 2, & 3, & 4, & \ldots, & n \end{pmatrix} (1, 2, 4, 5, \ldots, n+1)$$
$$= 0 \ldots\ldots\ldots (64).$$

In a similar manner we can eliminate all but the first four of our determinants, and we thus obtain an identity of the type

$$\begin{pmatrix} 1, & 5, & 6, & \ldots, & n+2 \\ 2, & 3, & 4, & \ldots, & n \end{pmatrix} (2, 3, 4, \ldots, n+1)$$
$$- \begin{pmatrix} 2, & 5, & 6, & \ldots, & n+2 \\ 2, & 3, & 4, & \ldots, & n \end{pmatrix} (1, 3, 4, \ldots, n+1)$$
$$+ \begin{pmatrix} 3, & 5, & 6, & \ldots, & n+2 \\ 2, & 3, & 4, & \ldots, & n \end{pmatrix} (1, 2, 4, 5, \ldots, n+1)$$
$$- \begin{pmatrix} 4, & 5, & 6, & \ldots, & n+2 \\ 2, & 3, & 4, & \ldots, & n \end{pmatrix} (1, 2, 3, 5, 6, \ldots, n+1)$$
$$= 0 \ldots\ldots\ldots (65).$$

We have also a set of equations of the type

$$\begin{pmatrix} 1, & 2 \\ 2, & 3 \end{pmatrix} (3, 4, 5, \ldots, n+2) - \begin{pmatrix} 1, & 3 \\ 2, & 3 \end{pmatrix} (2, 4, 5, \ldots, n+2)$$
$$+ \&c. + \begin{pmatrix} 2, & 3 \\ 2, & 3 \end{pmatrix} (1, 4, 5, \ldots, n+2) + \&c. = 0,$$

$$\begin{pmatrix} 1, & 2 \\ 2, & 4 \end{pmatrix} (3, 4, 5, \ldots, n+2) - \begin{pmatrix} 1, & 3 \\ 2, & 4 \end{pmatrix} (2, 4, 5, \ldots, n+2)$$
$$+ \&c. + \begin{pmatrix} 2, & 3 \\ 2, & 4 \end{pmatrix} (1, 4, 5, \ldots, n+2) + \&c. = 0,$$

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and so on. From these equations we readily deduce

$$\begin{pmatrix}1, 2, 6, 7, \ldots, n+2\\ 2, 3, 4, 5, \ldots, n\end{pmatrix}(3, 4, 5, \ldots, n+2)$$

$$-\begin{pmatrix}1, 3, 6, 7, \ldots, n+2\\ 2, 3, 4, 5, \ldots, n\end{pmatrix}(2, 4, 5, \ldots, n+2)$$

$$+\begin{pmatrix}1, 4, 6, 7, \ldots, n+2\\ 2, 3, 4, 5, \ldots, n\end{pmatrix}(2, 3, 5, 6, \ldots, n+2)$$

$$-\begin{pmatrix}1, 5, 6, 7, \ldots, n+2\\ 2, 3, 4, 5, \ldots, n\end{pmatrix}(2, 3, 4, 6, 7, \ldots, n+2)$$

$$+\begin{pmatrix}2, 3, 6, 7, \ldots, n+2\\ 2, 3, 4, 5, \ldots, n\end{pmatrix}(1, 4, 5, 6, \ldots, n+2)$$

$$-\begin{pmatrix}2, 4, 6, 7, \ldots, n+2\\ 2, 3, 4, 5, \ldots, n\end{pmatrix}(1, 3, 5, 6, \ldots, n+2)$$

$$+\begin{pmatrix}2, 5, 6, 7, \ldots, n+2\\ 2, 3, 4, 5, \ldots, n\end{pmatrix}(1, 3, 4, 6, 7, \ldots, n+2)$$

$$+\begin{pmatrix}3, 4, 6, 7, \ldots, n+2\\ 2, 3, 4, 5, \ldots, n\end{pmatrix}(1, 2, 5, 6, \ldots, n+2)$$

$$-\begin{pmatrix}3, 5, 6, 7, \ldots, n+2\\ 2, 3, 4, 5, \ldots, n\end{pmatrix}(1, 2, 4, 6, 7, \ldots, n+2)$$

$$+\begin{pmatrix}4, 5, 6, 7, \ldots, n+2\\ 2, 3, 4, 5, \ldots, n\end{pmatrix}(1, 2, 3, 6, 7, \ldots, n+2) = 0.$$

In virtue of the identities of the type (65), we see that the third, sixth, eighth, and tenth terms of this equation taken together come to zero, and that the fourth, seventh, and ninth terms taken together are equivalent to

$$-\begin{pmatrix}4, 5, 6, 7, \ldots, n+2\\ 2, 3, 4, 5, \ldots, n\end{pmatrix}(1, 2, 3, 4, 7, 8, \ldots, n+2).$$

The equation thus reduces to

$$\begin{pmatrix}1, 2, 6, 7, \ldots, n+2\\ 2, 3, 4, 5, \ldots, n\end{pmatrix}(3, 4, 5, \ldots, n+2)$$

$$-\begin{pmatrix}1, 3, 6, 7, \ldots, n+2\\ 2, 3, 4, 5, \ldots, n\end{pmatrix}(2, 4, 5, \ldots, n+2)$$

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$$+\begin{pmatrix}2,\ 3,\ 6,\ 7,\ \ldots,\ n+2\\ 2,\ 3,\ 4,\ 5,\ \ldots,\ n\end{pmatrix}(1,\ 4,\ 5,\ 6,\ \ldots,\ n+2)$$

$$-\begin{pmatrix}4,\ 5,\ 6,\ 7,\ \ldots,\ n+2\\ 2,\ 3,\ 4,\ 5,\ \ldots,\ n\end{pmatrix}(1,\ 2,\ 3,\ 4,\ 7,\ 8,\ \ldots,\ n+2)$$

$$=0\ldots\ldots\ldots(66).$$

We are now in a position to obtain expressions for the quantities of the type

$$\{r,\ 123\ldots(n+2)\}.$$

We have

$$\{123\ldots(n+1)\}=\frac{\partial}{\partial x_1}(2,\ 3,\ 4,\ \ldots,\ n+1)-\frac{\partial}{\partial x_2}(1,\ 3,\ 4,\ \ldots,\ n+1)$$

$$+\ldots+(-1)^n\frac{\partial}{\partial x_{n+1}}(1,\ 2,\ 3,\ \ldots,\ n)$$

$$=\begin{pmatrix}3,\ 4,\ 5,\ \ldots,\ n+1\\ 2,\ 3,\ 4,\ \ldots,\ n\end{pmatrix}[1,\ 12]-\begin{pmatrix}2,\ 4,\ 5,\ \ldots,\ n+1\\ 2,\ 3,\ 5,\ \ldots,\ n\end{pmatrix}[1,\ 13]$$

$$+\ldots+(-1)^{n-1}\begin{pmatrix}2,\ 3,\ 4,\ \ldots,\ n\\ 2,\ 3,\ 4,\ \ldots,\ n\end{pmatrix}[1,\ 1(n+1)]$$

$$+\begin{pmatrix}1,\ 4,\ 5,\ \ldots,\ n+1\\ 2,\ 3,\ 4,\ \ldots,\ n\end{pmatrix}[1,\ 23]-\begin{pmatrix}1,\ 3,\ 5,\ \ldots,\ n+1\\ 2,\ 3,\ 4,\ \ldots,\ n\end{pmatrix}[1,\ 24]$$

$$+\&c.+\begin{pmatrix}1,\ 2,\ 3,\ \ldots,\ n-1\\ 2,\ 3.\ 4,\ \ldots,\ n\end{pmatrix}[1,\ n(n+1)]$$

$$-\left\{\begin{pmatrix}3,\ 4,\ 5,\ \ldots,\ n+1\\ 1,\ 3,\ 4,\ \ldots,\ n\end{pmatrix}[2,\ 12]-\begin{pmatrix}2,\ 4,\ 5,\ \ldots,\ n+1\\ 1,\ 3,\ 4,\ \ldots,\ n\end{pmatrix}[2,\ 13]+\&c.\right\}$$

$$+\&c.$$

$$+(-1)^{n-1}\left\{\begin{pmatrix}3,4,5,\ldots,n+1\\ 1,2,3,\ldots,n-1\end{pmatrix}[n,12]-\begin{pmatrix}2,4,5,\ldots,n+1\\ 1,2,3,\ldots,n-1\end{pmatrix}[n,13]+\&c.\right\}.$$

Thus for the coefficient of $[s,\ pq]$ in the expression for

$$\{r,\ 123\ldots(n+2)\},$$

we obtain

$$(-1)^{s-1}\times(-1)^{q-p-1}\left\{a_{r1}\begin{pmatrix}2,3,4,\ldots,p-1,p+1,\ldots,q-1,q+1,\ldots,n+2\\ 1,2,3,\ \ldots\ldots\ldots,\ s-1,s+1,\ \ldots\ldots\ldots\ldots,n\end{pmatrix}\right.$$

$$\left.-a_{r2}\begin{pmatrix}1,\ 3,\ 4,\ \ldots,\ p-1,\ p+1,\ \ldots,\ q-1,\ q+1,\ \ldots,\ n+2\\ 1,\ 2,\ 3,\ \ldots\ldots\ldots,\ s-1,\ s+1,\ \ldots\ldots\ldots\ldots,\ n\end{pmatrix}+\&c.\right\}.$$

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Now, if s be different from r, this gives us a determinant of n rows and columns in which a row is repeated, and the determinant therefore vanishes.

Also the coefficient of $[r, pq]$ becomes

$$(-1)^{q-p-1}(1, 2, 3, \ldots, p-1, p+1, \ldots, q-1, q+1, \ldots, n+2),$$

and we thus obtain

$$\begin{aligned}\{r, 123\ldots(n+2)\} &= (3, 4, 5, \ldots, n+2)[r, 12] - (2, 4, 5, \ldots, n+2)[r, 13] \\ &+ \ldots + (-1)^{n+2}(2, 3, 4, \ldots, n+1)[r, 1(n+2)] \\ &+ (1, 4, 5, \ldots, n+2)[r, 23] - (1, 3, 5, 6, \ldots, n+2)[r, 24] \\ &+ \&c. + (1, 2, 3, \ldots, n)[r, (n+1)(n+2)].\end{aligned}$$

Now consider the expression

$$\begin{aligned}&(3, 4, 5, \ldots, n+2)\{1245\ldots(n+2)\} \\ &\quad - (2, 4, 5, \ldots, n+2)\{1345\ldots(n+2)\} \\ &\quad + (1, 4, 5, \ldots n+2)\{2345\ldots(n+2)\},\end{aligned}$$

which is similar in type to the coefficients of the various terms of equation (63). The coefficient of $[r, 1p]$ in this expression is

$$\begin{aligned}&(-1)^{r-1}(-1)^{p-1}\left\{(3, 4, 5, \ldots, n+2)\begin{pmatrix}2, 4, 5, \ldots, p-1, p+1, \ldots, n+2\\ 1, 2, 3, \ldots, r-1, r+1, \ldots, n\end{pmatrix}\right. \\ &\quad \left. - (2, 4, 5, \ldots, n+2)\begin{pmatrix}3, 4, 5, \ldots, p-1, p+1, \ldots, n+2\\ 1, 2, 3, \ldots, r-1, r+1, \ldots, n\end{pmatrix}\right\},\end{aligned}$$

which, on account of the identities of the type (64), becomes

$$\begin{aligned}&(-1)^{r-1}(-1)^{p}(2, 3, 4, \ldots, p-1, p+1, \ldots, n+2) \\ &\qquad \times \begin{pmatrix}4, 5, 6, \ldots\ldots\ldots\ldots, n+2\\ 1, 2, 3, \ldots, r-1, r+1, \ldots, n\end{pmatrix}.\end{aligned}$$

For the coefficient of $[r, pq]$ we obtain

$$\begin{aligned}&(-1)^{r-1}(-1)^{q-p-1} \\ &\times\left\{(3,4,5,\ldots,n+2)\begin{pmatrix}1,2,4,5,\ldots,p-1,p+1,\ldots,q-1,q+1,\ldots,n+2\\ 1,2,3,\ldots\ldots\ldots\ldots,r-1,r+1,\ldots\ldots\ldots\ldots,n\end{pmatrix}\right. \\ &-(2,4,5,\ldots,n+2)\begin{pmatrix}1,3,4,5,\ldots,p-1,p+1,\ldots,q-1,q+1,\ldots,n+2\\ 1,2,3,\ldots\ldots\ldots\ldots,r-1,r+1,\ldots\ldots\ldots\ldots,n\end{pmatrix} \\ &\left.+(1,4,5,\ldots,n+2)\begin{pmatrix}2,3,4,5,\ldots,p-1,p+1,\ldots,q-1,q+1,\ldots,n+2\\ 1,2,3,\ldots\ldots\ldots\ldots,r-1,r+1,\ldots\ldots\ldots\ldots,n\end{pmatrix}\right\},\end{aligned}$$

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which, in virtue of the identities of the type (66), reduces to

$$(-1)^{r-1}(-1)^{q-p-1}(1, 2, 3, \ldots, p-1, p+1, \ldots, q-1, q+1, \ldots, n+2) \times \begin{pmatrix} 4, 5, 6, \ldots\ldots\ldots\ldots, n+2 \\ 1, 2, 3, \ldots, r-1, r+1, \ldots, n \end{pmatrix}.$$

Thus we eventually obtain

$$\begin{aligned}(3, 4, 5, \ldots, n+2)\{1245\ldots(n+2)\} &- (2, 4, 5, \ldots, n+2)\{1345\ldots(n+2)\} \\ &+ (1, 4, 5, \ldots, n+2)\{2345\ldots(n+2)\} \\ &= \begin{pmatrix} 4, 5, 6, \ldots, (n+2) \\ 2, 3, 4, \ldots, n \end{pmatrix}\{1, 123\ldots(n+2)\} \\ &- \begin{pmatrix} 4, 5, 6, \ldots, n+2 \\ 1, 3, 4, \ldots, n \end{pmatrix}\{2, 123\ldots(n+2)\} \\ &\qquad + \&c. \\ + (-1)^{r-1}&\begin{pmatrix} 4, 5, 6, \ldots\ldots\ldots\ldots, n+2 \\ 1, 2, 3, \ldots, r-1, r+1, \ldots, n \end{pmatrix}\{r, 123\ldots(n+2)\} \\ + \&c. + (-1)^{n-1}&\begin{pmatrix} 4, 5, 6, \ldots, n+2 \\ 1, 2, 3, \ldots, n-1 \end{pmatrix}\{n, 123\ldots(n+2)\}.\end{aligned}$$

If we now write

$$\begin{aligned}L_r = \Sigma\frac{d(x_u, x_v)}{d(x_1, x_2)}&\{r, uv(k+1)(k+2)\ldots m\} \\ &- \Sigma\frac{dx_u}{dx_1}\{r, 2u(k+1)(k+2)\ldots m\} \\ &+ \Sigma\frac{dx_u}{dx_2}\{r, 1u(k+1)(k+2)\ldots m\} \\ &+ \{r, 12(k+1)(k+2)\ldots m\},\end{aligned}$$

we see by the above result that equation (63) may be written in the form

$$\begin{aligned}&\begin{pmatrix} k+2, k+3, \ldots, m \\ 2, 3, 4, \ldots, n \end{pmatrix}L_1 - \begin{pmatrix} k+2, k+3, \ldots, m \\ 1, 3, 4, \ldots, n \end{pmatrix}L_2 \\ &+ \&c. + (-1)^{r-1}\begin{pmatrix} k+2, k+3, \ldots\ldots\ldots, m \\ 1, 2, \ldots, r-1, r+1, \ldots, n \end{pmatrix}L_r \\ &+ \&c. + (-1)^{n-1}\begin{pmatrix} k+2, k+3, \ldots, m \\ 1, 2, 3, \ldots, n-1 \end{pmatrix}L_n = 0 \ldots (67).\end{aligned}$$

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Further, we shall obtain one equation of the type (67) from each of the n equations of the type

$$\frac{d^2x_s}{dx_1dx_2} = \frac{d^2x_s}{dx_2dx_1}$$

obtained by assigning to s the successive values

$$k+1,\ k+2,\ \ldots,\ m.$$

There being n of these equations, they will require either that the L's should vanish severally, or that the determinant of their coefficients should vanish. This determinant, however, is the reciprocal of the determinant

$$(k+1,\ k+2,\ \ldots,\ m),$$

and is therefore equivalent to

$$\{(k+1,\ k+2,\ \ldots,\ m)\}^{n-1}.$$

Thus, if we assume the determinant

$$(k+1,\ k+2,\ \ldots,\ m)$$

to be finite, we have

$$L_1 = L_2 = \ldots = L_r = \ldots = L_n = 0 \quad \ldots\ldots\ldots\ldots (68).$$

§ 32. We now proceed to develope the investigation further in some special cases. Take first the case in which the number of integrals is one more than the number of equations.

In this case the equation

$$L_r = 0$$

reduces to one of three terms, and assumes the form

$$\{r,\ 1k\ (k+1)\ (k+2)\ldots m\}\frac{dx_k}{dx_2} - \{r,\ 2k\ (k+1)\ (k+2)\ldots m\}\frac{dx_k}{dx_1} + \{r,\ 12\ (k+1)\ (k+2)\ldots m\} = 0 \ldots (69).$$

Writing, for brevity, this equation in the form

$$A_r\frac{dx_k}{dx_2} - B_r\frac{dx_k}{dx_1} + C_r = 0,$$

we see that the consistency of equations (68) will require a set of conditions of the type

$$(A_pB_qC_r) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (70).$$

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There are other conditions necessary besides these. Thus, if we take equation (69), which connects

$$\frac{dx_k}{dx_2} \text{ with } \frac{dx_k}{dx_1},$$

and combine it with the two corresponding equations that respectively connect

$$\frac{dx_k}{dx_2} \text{ with } \frac{dx_k}{dx_1}, \text{ and } \frac{dx_k}{dx_3} \text{ with } \frac{dx_k}{dx_2},$$

we see that the differential coefficients can be eliminated, and that we are left with the relation

$$\begin{aligned}&\{r, 12\,(k+1)\,(k+2)\,..m\}\,\{r, 3k\,(k+1)\,(k+2)...m\}\\ &-\{r, 13\,(k+1)\,(k+2)...m\}\,\{r, 2k\,(k+1)\,(k+2)...m\}\\ &+\{r, 23\,(k+1)\,(k+2)...m\}\,\{r, 1k\,(k+1)\,(k+2)...m\}\\ &\qquad\qquad = 0 \ldots\ldots\ldots\ldots (71).\end{aligned}$$

The conditions of the type (70) correspond to the conditions of the type (33) in article 22. Those of the type (71) are additional, and are analogous to those of the type (17) in article 18. They therefore enable us to effect a generalization of article 23 on similar lines to those in which the results of article 14 are generalized in articles 18 and 19.

Supposing the conditions of the type (70) to be satisfied, we see that it will only be necessary to consider a single pair out of equations (68) in the determination of

$$\frac{dx_k}{dx_1} \text{ and } \frac{dx_k}{dx_2}.$$

We will choose the equations

$$L_1 = 0 \text{ and } L_2 = 0.$$

Taking account of the form of L_r given in equation (69), we obtain for the solution of these equations

$$\frac{dx_k}{dx_1} = \frac{\begin{vmatrix} \{1,\ 1k\,(k+1)\,(k+2)...m\}, & \{2,\ 1k\,(k+1)\,(k+2)...m\} \\ \{1,\ 12\,(k+1)\,(k+2)...m\}, & \{2,\ 12\,(k+1)\,(k+2)...m\} \end{vmatrix}}{\begin{vmatrix} \{1,\ 1k\,(k+1)\,(k+2)...m\}, & \{2,\ 1k\,(k+1)\,(k+2)...m\} \\ \{1,\ 2k\,(k+1)\,(k+2)...m\}, & \{2,\ 2k\,(k+1)\,(k+2)...m\} \end{vmatrix}},$$

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and

$$\frac{dx_k}{dx_2} = \frac{\begin{vmatrix} \{1,\ 2k(k+1)(k+2)\ldots m\}, & \{2,\ 2k(k+1)(k+2)\ldots m\} \\ \{1,\ 12(k+1)(k+2)\ldots m\}, & \{2,\ 12(k+1)(k+2)\ldots m\} \end{vmatrix}}{\begin{vmatrix} \{1,\ 1k(k+1)(k+2)\ldots m\}, & \{2,\ 1k(k+1)(k+2)\ldots m\} \\ \{1,\ 2k(k+1)(k+2)\ldots m\}, & \{2,\ 2k(k+1)(k+2)\ldots m\} \end{vmatrix}}.$$

From the consideration of the equations of the type

$$\frac{d^2x_s}{dx_1 dx_2} = \frac{d^2x_s}{dx_2 dx_1}$$

we could obtain equations connecting

$$\frac{dx_k}{dx_1} \text{ with } \frac{dx_k}{dx_2},$$

and from the solution of these we should obtain

$$\frac{dx_k}{dx_2} = \frac{\begin{vmatrix} \{1,\ 3k(k+1)(k+2)\ldots m\}, & \{2,\ 3k(k+1)(k+2)\ldots m\} \\ \{1,\ 13(k+1)(k+2)\ldots m\}, & \{2,\ 13(k+1)(k+2)\ldots m\} \end{vmatrix}}{\begin{vmatrix} \{1,\ 1k(k+1)(k+2)\ldots m\}, & \{2,\ 1k(k+1)(k+2)\ldots m\} \\ \{1,\ 3k(k+1)(k+2)\ldots m\}, & \{2,\ 3k(k+1)(k+2)\ldots m\} \end{vmatrix}},$$

and a comparison of the value of

$$\frac{dx_k}{dx_1}$$

obtained from these equations with that given above reveals the existence of a set of conditions of the type

$$\begin{aligned} &\{r,\ 12(k+1)(k+2)\ldots m\}\{s,\ 3k(k+1)(k+2)\ldots m\} \\ &-\{r,\ 13(k+1)(k+2)\ldots m\}\{s,\ 2k(k+1)(k+2)\ldots m\} \\ &+\{r,\ 23(k+1)(k+2)\ldots m\}\{s,\ 1k(k+1)(k+2)\ldots m\} \\ &+\{s,\ 12(k+1)(k+2)\ldots m\}\{r,\ 3k(k+1)(k+2)\ldots m\} \\ &-\{s,\ 13(k+1)(k+2)\ldots m\}\{r,\ 2k(k+1)(k+2)\ldots m\} \\ &+\{s,\ 23(k+1)(k+2)\ldots m\}\{r,\ 1k(k+1)(k+2)\ldots m\} \\ &\qquad = 0 \ldots\ldots\ldots (72), \end{aligned}$$

which might be expected.

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Generally, we have

$$\frac{dx_k}{dx_u} = \frac{\begin{vmatrix} \{1, uk(k+1)(k+2)\ldots m\}, & \{2, uk(k+1)(k+2)\ldots m\} \\ \{1, 1u(k+1)(k+2)\ldots m\}, & \{2, 1u(k+1)(k+2)\ldots m\} \end{vmatrix}}{\begin{vmatrix} \{1, 1k(k+1)(k+2)\ldots m\}, & \{2, 1k(k+1)(k+2)\ldots m\} \\ \{1, uk(k+1)(k+2)\ldots m\}, & \{2, uk(k+1)(k+2)\ldots m\} \end{vmatrix}}.$$

If we take the equation

$$\{s, 1k(k+1)(k+2)\ldots m\}\frac{dx_k}{dx_2} - \{s, 2k(k+1)(k+2)\ldots m\}\frac{dx_k}{dx_1} + \{s, 12(k+1)(k+2)\ldots m\} = 0,$$

and combine it with the equations

$$(1, k+1, k+2, \ldots, r-1, r+1, \ldots, m) + (k, k+1, k+2, \ldots, r-1, r+1, \ldots, m)\frac{dx_k}{dx_1} + (-1)^{r-k-1}(k+1, k+2, \ldots, m)\frac{dx_r}{dx_1} = 0,$$

$$(2, k+1, k+2, \ldots, r-1, r+1, \ldots, m) + (k, k+1, k+2, \ldots, r-1, r+1, \ldots, m)\frac{dx_k}{dx_2} + (-1)^{r-k-1}(k+1, k+2, \ldots, m)\frac{dx_r}{dx_2} = 0,$$

we obtain

$$\begin{aligned}(-1)^{r-k}&(k+1, k+2, \ldots, m)\\ \times &\left[\{s, 1k(k+1)(k+2)\ldots m\}\frac{dx_r}{dx_2} - \{s, 2k(k+1)(k+2)\ldots m\}\frac{dx_r}{dx_1}\right]\\ &+ (1, k+1, k+2, \ldots, r-1, r+1, \ldots, m)\{s, 2k(k+1)(k+2)\ldots m\}\\ &- (2, k+1, k+2, \ldots, r-1, r+1, \ldots, m)\{s, 1k(k+1)(k+2)\ldots m\}\\ &+ (k, k+1, k+2, \ldots, r-1, r+1, \ldots, m)\{s, 12(k+1)(k+2)\ldots m\}\\ &= 0.\end{aligned}$$

Now we have a set of relations of the type

$$a_{u1}\{s, 2k(k+1)(k+2)\ldots m\} - a_{u2}\{s, 1k(k+1)(k+2)\ldots m\}$$

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$$+ a_{ik}\{s, 12(k+1)(k+2)\ldots m\} - a_{i(k+1)}\{s, 12k(k+2)(k+3)\ldots\}$$
$$+ \&c. + (-1)^{r-k} a_{ir}\{s, 12k(k+1)\ldots(r-1)(r+1)\ldots m\}$$
$$+ \&c. + (-1)^{n} a_{im}\{s, 12k(k+1)\ldots(m-1)\} = 0.$$

From these we readily deduce

$$(1, k+1, k+2, \ldots, r-1, r+1, \ldots, m)\{s, 2k(k+1)(k+2)\ldots m\}$$
$$= (2, k+1, k+2, \ldots, r-1, r+1, \ldots, m)\{s, 1k(k+1)(k+2)\ldots m\}$$
$$+ (k, k+1, k+2, \ldots, r-1, r+1, \ldots, m)\{s, 12(k+1)(k+2)\ldots m\}$$
$$= (k+1, k+2, \ldots, m)\{s, 12k(k+1)\ldots(r-1)(r+1)\ldots m\} = 0.$$

Thus our equation reduces to

$$(-1)^{r-k}\left[\{s, 1k(k+1)(k+2)\ldots m\}\frac{dx_r}{dx_2} - \{s, 2k(k+1)(k+2)\ldots m\}\frac{dx_r}{dx_1}\right]$$
$$+ \{s, 12k(k+1)\ldots(r-1)(r+1)\ldots m\} = 0.$$

It is thus clear that we shall have generally

$$(-1)^{r-k}\frac{dx_r}{dx_u} = \frac{\begin{vmatrix}\{1, uk(k+1)(k+2)\ldots\ldots m\}, \{2, uk(k+1)(k+2)\ldots\ldots m\}\\ \{1, 1uk(k+1)\ldots(r-1)(r+1)\ldots m\}, \{2, 1uk(k+1)\ldots(r-1)(r+1)\ldots m\}\end{vmatrix}}{\begin{vmatrix}\{1, 1k(k+1)(k+2)\ldots m\}, \{2, 1k(k+1)(k+2)\ldots m\}\\ \{1, uk(k+1)(k+2)\ldots m\}, \{2, uk(k+1)(k+2)\ldots m\}\end{vmatrix}},$$

Thus, corresponding to equations (34), we shall have a set of $n+1$ equations, of which the following is the typical equation

$$\frac{\begin{vmatrix}\{1, 1k(k+1)(k+2)\ldots\ldots m\}, \{2, 1k(k+1)(k+2)\ldots\ldots m\}\\ \{1, 12k(k+1)\ldots(r-1)(r+1)\ldots m\}, \{2, 12k(k+1)\ldots(r-1)(r+1)\ldots m\}\end{vmatrix}}{\begin{vmatrix}\{1, 1k(k+1)(k+2)\ldots m\}, \{2, 1k(k+1)(k+2)\ldots m\}\\ \{1, 2k(k+1)(k+2)\ldots m\}, \{2, 2k(k+1)(k+2)\ldots m\}\end{vmatrix}} dx_1$$
$$+ \frac{\begin{vmatrix}\{1, 2k(k+1)(k+2)\ldots\ldots m\}, \{2, 2k(k+1)(k+2)\ldots\ldots m\}\\ \{1, 12k(k+1)\ldots(r-1)(r+1)\ldots m\}, \{2, 12k(k+1)\ldots(r-1)(r+1)\ldots m\}\end{vmatrix}}{\begin{vmatrix}\{1, 1k(k+1)(k+2)\ldots m\}, \{2, 1k(k+1)(k+2)\ldots m\}\\ \{1, 2k(k+1)(k+2)\ldots m\}, \{2, 2k(k+1)(k+2)\ldots m\}\end{vmatrix}} dx_2$$
$$+ \&c.$$

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$$+\frac{\begin{vmatrix}\{1, uk(k+1)(k+2)\ldots\ldots m\}, & \{2, uk(k+1)(k+2)\ldots\ldots m\}\\ \{1, 1uk(k+1)\ldots(r-1)(r+1)\ldots m\}, & \{2, 1uk(k+1)\ldots(r-1)(r+1)\ldots m\}\end{vmatrix}}{\begin{vmatrix}\{1, 1k(k+1)(k+2)\ldots m\}, & \{2, 1k(k+1)(k+2)\ldots m\}\\ \{1, uk(k+1)(k+2)\ldots m\}, & \{2, uk(k+1)(k+2)\ldots m\}\end{vmatrix}} dx_u$$

$$+ \&c.$$

$$+\frac{\begin{vmatrix}\{1, (k-1)k(k+1)\ldots\ldots m\}, & \{2, (k-1)k(k+1)\ldots\ldots m\}\\ \{1, 1(k-1)k\ldots(r-1)(r+1)\ldots m\}, & \{2, 1(k-1)k\ldots(r-1)(r+1)\ldots m\}\end{vmatrix}}{\begin{vmatrix}\{1, 1k(k+1)(k+2)\ldots m\}, & \{2, 1k(k+1)(k+2)\ldots m\}\\ \{1, (k-1)k(k+1)\ldots m\}, & \{2, (k-1)k(k+1)\ldots m\}\end{vmatrix}} dx_{k-1}$$

$$+ (-1)^{r-k+1} dx_r = 0 \quad \ldots\ldots\ldots(73).$$

To complete the discussion of the necessary conditions, we should have to combine the values we have found for

$$\frac{dx_r}{dx_u}, \text{ \&c.,}$$

with the equations of the type

$$\frac{d^2x_r}{dx_u dx_v} = \frac{d^2x_r}{dx_v dx_u}.$$

At present, however, we are not so much concerned with the forms of the conditions as with the method of integration of our system of equations. It will be sufficient for our present purpose to remark that, whatever be the necessary conditions, they will secure that the set of $n+1$ equations of the type (73) shall constitute an exact system. Thus the $n+1$ integrals of our system of equations will be the $n+1$ common solutions of the system of equations, $k-1$ in number, whose typical equation is of the form

$$\begin{vmatrix}\{1, 1k(k+1)(k+2)\ldots m\}, & \{2, 1k(k+1)(k+2)\ldots m\}\\ \{1, uk(k+1)(k+2)\ldots m\}, & \{2, uk(k+1)(k+2)\ldots m\}\end{vmatrix} \frac{\partial\theta}{\partial x_u}$$

$$+\begin{vmatrix}\{1, uk(k+1)(k+2)\ldots m\}, & \{2, uk(k+1)(k+2)\ldots m\}\\ \{1, 1u(k+1)(k+2)\ldots m\}, & \{2, 1u(k+1)(k+2)\ldots m\}\end{vmatrix} \frac{\partial\theta}{\partial x_k}$$

$$-\begin{vmatrix}\{1, uk(k+1)(k+2)\ldots m\}, & \{2, uk(k+1)(k+2)\ldots m\}\\ \{1, 1uk(k+2)(k+3)\ldots m\}, & \{2, 1uk(k+2)(k+3)\ldots m\}\end{vmatrix} \frac{\partial\theta}{\partial x_{k+1}}$$

$$+ \&c.$$

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$$+(-1)^{r-k}\begin{vmatrix} \{1,\ uk\,(k+1)\,(k+2)\ldots m\},\ \{2,\ uk\,(k+1)\,(k+2)\ldots m\} \\ \{1, 1uk\,(k+1)\ldots(r-1)\,(r+1)\ldots m\}, \\ \qquad\qquad \{2,\ 1uk\,(k+1)\ldots(r-1)\,(r+1)\ldots m\} \end{vmatrix} \frac{\partial\theta}{\partial x_r}$$

$$+ \&c.$$

$$+(-1)^{n}\begin{vmatrix} \{1,\ uk(k+1)(k+2)\ldots m\},\ \{2,\ uk\,(k+1)\,(k+2)\ldots m\} \\ \{1, 1uk\,(k+1)\ldots..(m-1)\},\ \{2, 1uk\,(k+1)\ldots..(m-1)\} \end{vmatrix} \frac{\partial\theta}{\partial x_n}$$

$$= 0.$$

§ 33. As the excess of the number of integrals in our original system of equations above the number of equations in that system increases, so does the algebraical treatment of the equations of the type (68) become the more complicated. The general treatment of these equations will undoubtedly require the full development of a suitable set of derived functions, as briefly sketched in Part IV. We can, however, shew how equations (68) may be reduced when the number of integrals is $n+2$. In this case the equation

$$L_r = 0$$

takes the form

$$\begin{aligned} &\frac{d(x_{k-1},\ x_k)}{d(x_1,\ x_2)}\{r,\ (k-1)\,k\,(k+1)\ldots m\} \\ &\quad + \frac{dx_{k-1}}{dx_2}\{r,\ 1\,(k-1)\,(k+1)\,(k+2)\ldots m\} \\ &\quad + \frac{dx_k}{dx_2}\ \{r,\ 1k\,(k+1)\,(k+2)\ldots m\} \\ &\quad - \frac{dx_{k-1}}{dx_1}\{r,\ 2\,(k-1)\,(k+1)\,(k+2)\ldots m\} \\ &\quad - \frac{dx_k}{dx_1}\ \{r,\ 2k\,(k+1)\,(k+2)\ldots m\} \\ &\qquad + \{r,\ 12\,(k+1)\,(k+2)\ldots m\} = 0 \ldots (74). \end{aligned}$$

Denoting, for brevity, the left-hand side of equation (74) by the symbol (1, 2), we proceed to form the equation

$$(2,\ 3)\frac{dx_{k-1}}{dx_1} - (1,\ 3)\frac{dx_{k-1}}{dx_2} + (1,\ 2)\frac{dx_{k-1}}{dx_3} = 0.$$

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We obtain

$$\begin{aligned}
&\frac{d(x_{k-1}, x_k)}{d(x_2, x_3)}\{r, 1k(k+1)(k+2)\ldots m\} \\
&\quad - \frac{d(x_{k-1}, x_k)}{d(x_1, x_3)}\{r, 2k(k+1)(k+2)\ldots m\} \\
&\quad + \frac{d(x_{k-1}, x_k)}{d(x_1, x_2)}\{r, 3k(k+1)(k+2)\ldots m\} \\
&\quad = \{r, 23(k+1)(k+2)\ldots m\}\frac{dx_{k-1}}{dx_1} \\
&\quad - \{r, 13(k+1)(k+2)\ldots m\}\frac{dx_{k-1}}{dx_2} \\
&\quad + \{r, 12(k+1)(k+2)\ldots m\}\frac{dx_{k-1}}{dx_3},
\end{aligned}$$

Combining this equation with the three equations of the type (74) from which it was deduced, we obtain

$$\begin{aligned}
&\frac{dx_{k-1}}{dx_3}[\{r, 1k(k+1)(k+2)\ldots m\}\{r, 2(k-1)(k+1)(k+2)\ldots m\} \\
&\quad - \{r, 1(k-1)(k+1)(k+2)\ldots m\}\{r, 2k(k+1)(k+2)\ldots m\} \\
&\quad + \{r, 12(k+1)(k+2)\ldots m\}\{r, (k-1)k(k+1)\ldots m\}] \\
&- \frac{dx_{k-1}}{dx_2}[\{r, 1k(k+1)(k+2)\ldots m\}\{r, 3(k-1)(k+1)(k+2)\ldots m\} \\
&\quad - \{r, 1(k-1)(k+1)(k+2)\ldots m\}\{r, 3k(k+1)(k+2)\ldots m\} \\
&\quad + \{r, 13(k+1)(k+2)\ldots m\}\{r, (k-1)k(k+1)\ldots m\}] \\
&+ \frac{dx_{k-1}}{dx_1}[\{r, 2k(k+1)(k+2)\ldots m\}\{r, 3(k-1)(k+1)(k+2)\ldots m\} \\
&\quad - \{r, 2(k-1)(k+1)(k+2)\ldots m\}\{r, 3k(k+1)(k+2)\ldots m\} \\
&\quad + \{r, 23(k+1)(k+2)\ldots m\}\{r, (k-1)k(k+1)\ldots m\}] \\
&- [\{r, 1k(k+1)(k+2)\ldots m\}\{r, 23(k+1)(k+2)\ldots m\} \\
&\quad - \{r, 2k(k+1)(k+2)\ldots m\}\{r, 13(k+1)(k+2)\ldots m\} \\
&\quad + \{r, 3k(k+1)(k+2)\ldots m\}\{r, 12(k+1)(k+2)\ldots m\}] \\
&\qquad = 0 \quad \ldots\ldots\ldots (75).
\end{aligned}$$

We can also deduce a similar equation for the differential coefficients of x_k. These equations being linear may be subjected to the treatment of the preceding article. The similarity of the coefficients of equation (75) to the left-hand side of equation (71) will be noted.

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ON EXTENDED HIGH FACTORISATIONS.

(*Principally with reference to the form* $N = y^2 + 1$).

By H. J. Woodall, A.R.C.Sc.

[The Author is much indebted to Lt.-Col. Cunningham, R.E., for help in the preparation of this paper.]

TABLES of the factorisation of $y^2 + 1$ and of the congruence-solutions of

$$y^2 + 1 \equiv 0 \pmod{p},$$

as prepared by Col. Cunningham for his paper "On High Primes, $p = 4\varpi + 1$, $6\varpi + 1$, and factorisations," (pp. 10–21, of the present number of the *Quarterly Journal*) lead, very simply, to a method of "generation" of high primes which is quite independent of any assistance to be obtained by factor-tables of any other form. The primes thus obtained do not appear consecutively; but, if the work be carried far enough, we may be certain that every prime of the appropriate form will, in time, be thus disclosed.

In Col. Cunningham's paper it is shown that $y^2 + 1$ can be factorised very easily indeed, provided y is not too great. Also that the complete factorisation of this table up to a limit $N = y_m^2 + 1$ provides solutions of

$$y^2 + 1 \equiv 0 \pmod{p}$$

for all primes $p = 4\varpi + 1$ *continuously* up to the limit $p \not> 2y_m$, and also for a great many more primes $p = 4\varpi + 1$ $(> 2y_m)$.

It is the purpose of this note to prove that these "great many more primes $> 2y_m$" suffice for the complete factorisation of *all* numbers

$$N = y^2 + 1$$

up to the limit $N = (3y_m^2) + 1$.

Proof. Let $$y_r^2 + 1 = p.q,$$

where $$p < y_r < q,$$

therefore $$(y_r - p)^2 + 1 \equiv 0 \pmod{p}.$$

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But, since y_r is $\not> 3y_m$,

therefore p is $\not> 3y_m$,

divide the range of p into two parts

(α) $1 < p < 2y_m$,

(β) $2y_m < p < 3y_m$.

In case (α), we have seen that the congruence solutions for p are already known by reason of the completion of the table y^2+1 up to limit $N = y_m^2 + 1$.

(β) Since $2y_m < p < 3y_m$,

therefore $y_r - 2y_m > y_r - p > y_r - 3y_m$,

but $y_r \not> 3y_m$,

therefore *a fortiori* $y_r - p < y_m$,

therefore factors of $(y_r - p)^2 + 1$

are known because the factorisation table is complete to $(y_m^2 + 1)$.

Therefore, if $y_r^2 + 1$ is factorisable, its lesser factor (or factors) will be known provided $y_r \not> 3y_m$, where $y_m^2 + 1$ is the present limit of complete factorisation of the form

$$N = y^2 + 1.$$

Having completed the factorisation of all numbers

$$N = y^2 + 1$$

up to a limit $N = y_m^2 + 1$,

and having entered *all* the congruence-solutions $(y_r, p - y_r)$ thus found of

$$y^2 + 1 \equiv 0 \pmod{p}$$

into a second table. Next enter p against every y (in the first table), such that

$$y = y_r + kp,$$

or $$= (p - y_r) + k'p$$

(k and k' having all values from zero up to the limits of the table, *i.e.* up to $y_r \not> 3y_m$).

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Perform this operation for *every* p for which any entry has been made in the table of congruence-solutions.

Next, perform all the indicated divisions for all

$$N = y_r^2 + 1 \text{ (where } y_r \not> 3y_m),$$

and note that where the residual factor of such N is not less than the square of the least of the factors previously found, it will be necessary to test this residual factor by division by each of these known prime factors, repeating the division as often as necessary. When such test is successful there will be thus found a solution of the congruence

$$y^2 + 1 \equiv 0 \pmod{p^k}$$

with the appropriate value of k. And all such solutions should be tabulated.

Next, all residual factors for which we have thus failed to find factors must be primes.

[*Proof.* For, if such a number had a factor, the lesser factor, being $< y_r < 3y_m$, would have been discovered as previously shown].

These residual factors may therefore be entered as primes in the table of congruence-solutions provided $y_r \not> 3y_m$.

By this work we have extended the limit up to which all numbers $N = y^2 + 1$ are completely factorised, to $N = y_{3m}^2 + 1$.

Writing y_{3m} for $3y_m$ the process, described above, can now be repeated, substituting

$$3m \text{ for } m.$$

All this, as Col. Cunningham points out to the writer, is equally true for

$$N = y^2 \pm y + 1$$

with the primes $p \equiv 6\varpi + 1$.

As an example it follows at once that Col. Cunningham's "Factorisation Table of (y^2+1) being complete to (5000^2+1) "suffices to give complete factorisation to (15000^2+1), *i.e.* to "$2\frac{1}{4}$ hundred millions.

"Similarly the Table of $(y^3 \mp 1) \div (y \mp 1)$ factorisation of which "form is at present complete only to $(4000^3 \mp 1) \div (4000 \mp 1)$ "would give already complete factorisation to $(12000^3 \mp 1)$ "$\div (12000 \mp 1)$, *i.e.* to 144 million."*

It is scarcely necessary to point out that, although this

[* Extract from letter from Col. Cunningham to the writer.]

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method furnishes all the high primes of specified form, they come to hand in anything but a consecutive manner. In fact

first the array y^2+1,
second „ $\frac{1}{2}(y^2+1)$,
third „ $\frac{1}{3}(y^2+1)$,

and so forth.

The best form for collecting the primes as they occur is one in double columns, the headings being the consecutive thousands, the remaining three figures coming under p in the first half—the second half being reserved for the least solution (y_r) of the congruence

$$y^2+1 \equiv 0 \pmod{p},$$

e.g. $p = 9465601 = \frac{1}{2}(4351^2+1)$

would be entered thus

9464...		9465...		9466...	
p	y_r	p	y_r	p	y_r
		601	4351		

Note. These results may be generalised. The form $y^2 \mp a$ being the most general amongst quadratics.

We need not consider $ay^2+2by+c$, because the linear transformation $(ay+b)=y_1$ reduces this to $(y_1^2+ac-b^2) \div a$.

Consider now the form

$$y^2 \mp a,$$

of which there are three cases:

(1) a square $= b^2$, say, with the upper sign,

(2) a square $= b^2$, say, with the lower sign,

(3) a non-square.

(1) The congruence-solutions for this case are easily obtained from those of

$$y^2+1 \equiv 0 \pmod{p} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let y_r be a solution of (1), then ay_r is a solution of

$$y^2+b^2 \equiv 0 \pmod{p} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

provided b does not contain p. If b contains p, then zero is a solution of

$$y^2+b^2 \equiv 0, \pmod{p}.$$

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In order, therefore, to form a factorisation table of $N = y^2 + b^2$, it is only necessary to multiply the corresponding solutions (y_r and $p - y_r$) of

$$y^2 + 1 \equiv 0 \pmod{p}$$

by b and make the proper entries, *i.e.*, p will be a factor of

$$y^2 + b^2,$$

whenever $y \equiv by_r \pmod{p}$,

or $\equiv b(p - y_r) \pmod{p}$.

If, however, the congruence-solution table of $y^2 + 1$ had not been previously obtained, it is evidently possible to make a fresh one on the basis of

$$N = y^2 + b^2,$$

as explained in Col. Cunningham's paper. Gauss [*Werke*, Band II., pp. 478–496] has such tables of $N = y^2 + b^2$ ($b = 1, 2$, up to 9). [His factorisations extend to *very high* numbers, but only include such as are wholly made up of factors $\not> 200$].

(2) $y^2 - b^2 = (y - b)(y + b)$ and need not be considered further.

(3) The form $N = y^2 \mp a$ would give difficulty if we looked at it from the point of view of the form of its divisors. In that case we should need (Mathews, § 46).

$$\text{for } N = y^2 + a \text{ to solve } \left(\frac{-a}{p}\right)_2 = +1,$$

$$\text{,, } N = y^2 - a \text{ to solve } \left(\frac{a}{p}\right)_2 = +1.$$

But this is *not necessary*. We only need to calculate the numerical value of $y^2 \pm a$ (as the case may be) for each value of y from 1 to y_m (say), and obtain the numerical factors by very easy arithmetical work—next, enter *all* of these in a *new* table of congruence-solutions. Then, from Col. Cunningham's theorem, we know that no prime (*possible to the form*) less than $2y_m$ has been omitted.

The theorem in this note does not come into operation until y_m exceeds a certain value depending upon the magnitude of a. To find this minimum value of y_m, put

$$y_r^2 + a = pq \text{ (where } p < q, \text{ and } y_r < y_{3m}),$$

then $$(y_r \sim p)^2 + a \equiv 0 \pmod{p}.$$

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This form will have been included amongst those completely factorised if

$$y_r \sim p < y_m \ (y_r < y_{2m}).$$

(*a*) This is obvious if $y_r > p$, as the conditions reduce to $y_{2m} > p$.

(*b*) Consider $p - y_r < y_m$,

but $$p < \sqrt{(y_r^2 + a)} < q,$$

therefore $$\sqrt{(y_r^2 + a)} < y_r + y_m < 4y_m,$$

therefore $y_{2m}^2 + a < (4y_m)^2$, therefore $9y_m^2 + a < 16y_m^2$.

Hence the theorem of this paper holds:—That when

$$y_m > \sqrt{(\tfrac{1}{7}a)},$$

a table of the factorisation of

$$N = y^2 + a$$

complete for all values of y from y_1 to y_m will furnish material sufficient for the complete factorisation of

$$N = y^2 + a$$

up to the limit $N = (3y_m)^2 + a$. Also a may be positive or negative.

It is not difficult to prove that if a contains a square factor (say $a = b^2c$), that square factor may be rejected if the sole object is to determine the primes which will occur in the factor-table of

$$N = y^2 + a.$$

For, if we consider the form

$$N_1 = y_1^2 + c,$$

we have at once

$$b^2N_1 = (by_1)^2 + b^2c = b^2y_1^2 + a.$$

And this latter form will cover the whole of the ground of the former.

Turning our attention to powers higher than the second. We note that this method is not generally practicable. The condition being that there must be some relation, connecting the quantities in the form, which will reduce it to a quadratic.

When the power is the third, the constant must be a complete cube, *e.g.*

$$N = (y^3 \mp 1) \div (y \mp 1),$$

which is $y^2 \pm y + 1$ (a quadratic).

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This might, however, be treated thus

$$4N = 4(y^2 \pm y + 1) = (2y \pm 1)^2 + 3,$$

a special form of

$$N = y^2 + a.$$

It is obvious that if a number N be resolved into the form $y_t^2 + a$, it would be possible to factorise that number (or prove it prime) by the methods of this paper, *i.e.* by working on the form $N = y^2 + a$ and factorising, as shown, for every consecutive value of y from $y = 1$ to $y = y_t$.

NOTE BY LIEUT.-COL. CUNNINGHAM, R.E. (*Extended Factorisation*)—It is important to show how far the congruence-solutions must be tabulated beyond the limit $p \not< 2y_m$ (up to which the table is supposed complete) to ensure factorisation and detection of high primes up to the limit

$$N = (3y_m)^2 + 1.$$

Numbers $N < (2y_m)^2 + 1$, and also numbers $N > (2y_m)^2 + 1$, but $< (3y_m^2) + 1$, containing a factor $p \not> 2y_m$, can be completely factorised by the use of factors $p \not> 2y_m$. Numbers

$$N > (2y_m)^2 + 1, \text{ and } < (3y_m)^2 + 1,$$

containing no factors $p < 2y_m$ alone need the extended congruence table ($p > 2y_m$). If such numbers be *composite*, they contain at most two factors, say $N = pq$, $(p < q)$.

Here $p > 2y_m$, but $< \sqrt{N}$, *i.e.* $< 3y_m$,

$q > p$, and $> \sqrt{N}$, *i.e.* $> 3y_m$, but $\not> N \div p$,

i.e. $\not> \{(3y_m^2) + 1\} \div p$, *i.e.* $q \not> 4\frac{1}{2}y_m$.

Thus it suffices to extract from the factorisation-table of $N = y^2 + 1$ (supposed *complete* to the limit y_m) the roots of this congruence $y^2 + 1 \equiv 0 \pmod{p}$ for *all* values of $p \not> 2y_m$ (thus completing the congruence-solution table to that limit) and for all *additional* values of p (or q) therein occurring up to the limit p (or q) $\not> 4\frac{1}{2}y_m$ to ensure factorisation and detection of high primes up to the limit of $N = (3y_m)^2 + 1$.

A precisely similar argument applies to numbers

$$N = y^2 \mp y + 1.$$

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ON THE ANALYSIS OF LINEAR SETS OF POINTS.

By W. H. Young.

Introductory.

§ 1. AS far as I know, the subject of the analysis of sets of points in general has only been treated by Cantor in his paper in *Acta Math.* 7. The subject falls into two divisions, (1) the analysis of closed sets, (2) that of open sets. The former, which had already been attacked in parts by Bendixon, and fully developed by Cantor in *Annalen* 23, has become a recognised section of the theory of sets of points, and has been continually applied by writers on the subject. The latter has remained buried.

Schoenflies, who, in his *Bericht*, pp. 71–3, gives a brief sketch of Cantor's reasoning and results, makes, in conclusion, the remark that so far no application of the latter had yet been made. The reasons for this may have been twofold:—(1) the theory of open sets has so far been little investigated, and, (2) Cantor's reasoning is exceedingly abstruse and presupposes familiarity with the theory of his transfinite ordinal numbers. Cantor's point of view, indeed, seems to have been, partly at least, that of illustrating the use of his numbers. In both cases, (closed and open sets), he carries out a series of operations having the properties of a well-ordered set, (viz. every operation has a next one following it, but not always a next preceding it), and he discusses what remains of the set after these operations have all been carried out, with a view to proving that only transfinite numbers of his first and second classes are required to characterise the order of this series of operations, and the place to be assigned in it to the final operation. The properties of the partial sets into which the original set is thus analysed only occupy him in so far as they bear on this main question.

In the ordinary applications of the Theory of Sets of Points, however, it is usually not the process by which this analysis of Cantor's has been carried out, but the analysis itself, that is important. Theorem 5 of the present paper, indeed, is one of the recognised fundamental theorems required in all applications of the theory; with regard to Theorem 3, it is probable that its position with regard to the theory of open sets is not less fundamental than that of Theorem 5 in the theory of

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closed sets. The ideas embodied in these theorems, however, as well as the proofs of them, do not in any way necessitate much previous knowledge. The present paper is an attempt to give a systematic and yet elementary account of the matter. The analysis of a set into a part which is essential, and another which for many purposes is unimportant, comes first, and the theorems which in Cantor's work occupy the more conspicuous position follow. We then proceed to discuss, with illustrative examples, the mutual properties of the "adherences" and "coherences" introduced at the various stages of Cantor's process of dealing with an open set, to one another, and to the corresponding elements of the set obtained by closing the original set. No account of this part of the subject, so far as I know, exists, and in this connection I give the enunciation and proof of a theorem to which I have myself been led in the further development of the theory of open sets.

In no part of the present paper is any use made of Cantor's numbers.

Points L'.

§ 2. *Definition.* If L be a limiting point and d any interval, containing L as internal point, then, if d contains only a countable number of points of the given set, the same will be true when we diminish d as much as we please, as long as it contains L; in this case L will be said to be *a limiting point of countable degree.* If this is not the case L will be said to be of *more than countable degree.*

If a limiting point of more than countable degree be also a point of the set considered, we shall for shortness speak of it as *a point L'.*

With this explanation we are now able to state the following preliminary theorems.

THEOREM 1. *A set is countable, or not, according as it has not, or has, a point L'.*

For let (A, B) be the segment in which we are operating. Suppose that there is a point L'. Then it is evident that the set is more than countable.

Assume on the other hand that there is no point L'. Then taking any point P of the given set E, we can determine an interval d_1, having P as internal point and containing only a countably infinite, or a finite, number of other points of E. If outside d_1 there is any point of E left, we can in like manner

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describe about it an interval d_2, containing at most a countably infinite number of points of E, and so on.

Proceeding in this way, we shall be able eventually to shut up all the points of the set in such intervals. Now, by a well-known theorem, since the intervals may clearly be chosen to be non-overlapping, they are countable in number. Hence, since each interval contains only a countable number of points, it follows from the fact that a countable number of countable points is countable, that the set of points E is countable.

Cor. Those points of E which are not points L' are at most countable.

THEOREM 2. *If a set E has at least one point L', it has a more than countable set of them; and the set of such points L' is dense in itself*; (that is, every point is a limiting point for the set formed by them).

For consider the set consisting of E without the set of points L'; this must, by Theorem 1, be countable. Hence, if the points L' are themselves countable, the original set, being the sum of two countable sets, is itself countable, which is impossible if it had one point L'.

It remains to prove that the set of points L' is dense in itself.

Taking any one point L', there must either in (A, L'), or in (L', B), or in both, be a more than countable number of points of E. Suppose that this is the case in (A, L'). Divide up the segment (A, L') as follows. Bisect it at A_1, then bisect (A_1, L') at A_2, then (A_2, L') at A_3, and so on. We thus get a countably infinite set of intervals $(A, A_1), (A_1, A_2), \ldots$ such that no last interval exists, and such that L' is the sole external point in the segment (A, L').

In one at least of these intervals there must be a more than countable number of points of E, otherwise, as before, there would only be a countable number of points of E in (A, L'). Let (A_i, A_{i+1}) be such a segment, then by the preceding theorem there must be a point L' in it.

Treating the segment (A_{i+1}, L') as we did the segment (A, L'), it follows, from the definition of a point L', that it must contain a second point L_2', lying, say, in the interval (A_j, A_{j+1}). We next take the segment (A_{j+1}, L'), and so on. In this way we obtain a whole sequence of points L', having as limit that point L' with which we originally started. Thus any point L' is a limit for points of the same kind, that is to say, the set of points L' is dense in itself. Q.E.D.

These results may be summed up in the following manner:—

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THEOREM 3. Of the Nucleus.

Every set which is more than countable has a component, called the Nucleus, consisting of all its points L'. The Nucleus is dense in itself and more than countable, while the points of the set which are not contained in it are at most countably infinite.

Cor. Every set which contains no component dense in itself is countable.

This, which is true for all sets, is the fundamental theorem in the case of an open set; in the case of a closed set we can go a step further.

Closed and perfect sets.

§ 3. A set which is closed and dense in itself is called a perfect set.

By a theorem, due to Cantor and easily proved, a perfect set is more than countable, and has, indeed, the potency of the continuum.

Hence we can prove the following theorem:—

THEOREM 4. *A perfect set is its own Nucleus.*

For, if P be any point of a perfect set E, then inside any interval containing P there are points of E, which, with possibly the two end points, evidently form a closed and therefore a perfect set, and, as such, are more than countable. Thus P is a point L', which proves the theorem.

Cor. Every perfect set contained in a given set is contained in its Nucleus.

THEOREM 5. *The Nucleus of a closed set is a perfect set, and contains every component of E which is dense in itself.*

For it is clear that any limiting point of points L' is itself a point such that, in any segment, however small, containing it, there is a more than countable set of points of E; also, since E is closed, each limiting point is a point of E, and therefore, by definition, a point L'. Thus the Nucleus is a perfect set.

Further, any component of E dense in itself is contained in the perfect set got by closing it, and, since E is closed, this latter is contained in E, and therefore, by the preceding corollary, in the Nucleus. Q.E.D.

Derived and deduced sets.

§ 4. We now proceed to show how a closed set may, by means of the familiar processes of derivation and deduction, be analysed into its countable part and its Nucleus.

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By the process of *deduction* we mean that of taking all the points common to all the sets of a given infinite series of sets of points. If the sets are closed, and each a component of all its predecessors, as is the case with the successive derived sets of a given set, it is easily proved that at least one point can be obtained by deduction. Thus, if the derived sets of every order exist, a set may be deduced from them which we call the first deduced set, and which, like the derived sets themselves, is (1) closed and (2) contained in all its predecessors, viz. in all the derived sets.

Subjecting this set, and all sets and sequences of sets subsequently obtained, to the processes of derivation and deduction, we continue obtaining sets with these two properties, which are called the derived and deduced sets of the given set. The series can only come to an end if we ever arrive at a set consisting of a finite number of points only. In the contrary case no two of the sets will be identical, unless we ever arrive at a perfect set, and then, from that point on, they will all be identical.

THEOREM 6. *All derived and deduced sets have the same Nucleus.*

For, by derivation and deduction, a perfect set is unaltered; therefore, denoting by E^* the Nucleus of the first derived set E', E^* will be contained in every derived and deduced set, and therefore in the Nucleus of any one of them. That E^* must be identical with this latter set follows, since any derived or deduced set is a component[1] of E', so that any point L' of the former is a point L' of E'.

Cor. If E be closed, E^* is the Nucleus of E itself.

THEOREM 7. *The set of derived and deduced sets which are distinct from one another is at most countably infinite.*

For, as has been remarked, no two can be identical unless they are perfect, in which case, by Theorem 4, each would be its own Nucleus, that is E^*, and no further process of derivation or deduction would introduce a new set. Hence, as long as the sets are distinct, we can assign one point of E', not belonging to E^*, to each, and insure that no two sets shall correspond to the same point. That is, we set up a (1, 1)-correspondence between the different derived and deduced sets and points of a set, which, by Theorems 3 and 5, is known to be countable, therefore these sets themselves can be arranged in countable order.

[1] i. e. a set contained in another.

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Now as long as we do not arrive at a finite number of points, the series of derived and deduced sets cannot, as already pointed out, leave off. If the first derived set E' be more than countable, this cannot be the case, since E^*, being perfect, is, by Theorem 6, always present. In this case, then, we must, after a countable series of operations, arrive at a perfect set containing E^*, and, by Theorem 5, contained in E^*, that is, we arrive at E^* itself. Summing up, we have the following theorem and corollary.

THEOREM 8. *Every closed countable set can be reduced after a countable series of derivations and deductions to a finite number of points, and every more than countable closed set to its Nucleus.*

Cor. Every closed set is either countable or has the potency c.

For unclosed sets the theorem corresponding to Theorem 8, which has been also proved in the course of the preceding paragraph, may be stated as follows.

THEOREM 9. *Every set, whose first derived is countable, leads after a countable series of derivations and deductions to a finite number of points, and every set, whose first derived is more than countable, to a perfect set E^*.*

E^* is the Nucleus of any derived or deduced set of E, and contains every component dense in itself of E, in particular the Nucleus of E.

This latter theorem, which for some purposes is sufficient, does not give us, as in the case of Theorem 8, an analysis of the given set into components, since neither the finite number of points in the former case, nor the set E^* in the latter case, are necessarily components of E. Nor in the latter case does it enable us to draw any precise conclusion as to the potency of an open set.

In the following section we proceed to develop the method devised by Cantor for analysing open sets in a manner precisely analogous to that used above for closed sets.

Adherences and coherences.

§ 5. The term "adherent" has been adopted by Cantor to denote any isolated point of a set, and "coherent" to denote any limiting point which is also a point of the set; the set of all the adherents he calls the *adherence*, and that of all

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the coherents the *coherence*, and denote these symbolically by the addition of a small a or c respectively to the symbol used for the set itself; thus

$$E = Ea + Ec.$$

The adherence Ea is thus an isolated and therefore a countable set, while the coherence Ec consists of limiting points of Ea and possibly limiting points of Ec; thus if the set E is not dense in itself, there is at least one adherent, and every component of E which is dense in itself is a component of Ec. Also Ec belongs to the first derived set E', and Ea does not. If, and only if, E is closed, Ec is the same as E'. It will for some purposes be convenient to speak of Ec as the *first derived coherence*.

If there be no second derived set, Ec will consist solely of its own adherents, and the process comes to an end. Also if Ec be dense in itself, it consists solely of its own coherents, and the process cannot, by repetition, lead to any new set. If, however, there be a second derived set, and Ec be neither an isolated set, nor dense in itself, it will possess an adherence and a coherence, and we may proceed in our analysis a stage further,

$$Ec = Eca + Ecc;$$

whence

$$E = Ecc + Ea + Eca.$$

Here Ecc is contained in Ec', the first derived of Ec, but Eca is not. Thus Ecc is contained in E'', the second derived of E, and we shall therefore sometimes speak of it as the *second derived coherence*; it does not, however, follow that Eca is not contained in E'', or that it has not at least some points common with E'', or with some subsequent derived or deduced set; all we know is that Eca certainly is contained in E', and we shall therefore sometimes speak of it as the *first derived adherence*.

Ex. 1. Take the segment (0, 1), bisect it, and then bisect the right-hand part, and then bisect the new right-hand part, and so on, *ad inf.* The set formed by the points of division we denote as the set T, in the segment (0, 1); in the binary notation its points are

$$0,\quad 1,\ .1,\ .11,\ .111,\ \ldots.$$

This is the standard example (Cantor's) of a countable set whose first derived consists of a single point, viz. the point 1.

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A set T_2, the standard example of a countable set whose second derived consist of a single point, is constructed by inserting, between each pair of consecutive points of a set T_1, a set T_1.

Similarly a set T_n is constructed, by inserting between each pair of consecutive points of a set T_{n-1} a set T_1, and forms the stock example of a countable set, whose n^{th} derived consists of a single point.

Having premised this, we construct our example by taking the set T_2 in the segment (0, 1), omitting all the limiting points of the first order, that is all the points

$$.01, \quad .011, \quad .0111, \quad \ldots,$$

$$.101, \quad .1011, \quad .10111, \quad \ldots,$$

$$.1101, \; .11011, \; .110111, \; \ldots,$$

and so on ; generally omitting every point $.1^m01^n$.

Here *Ec* consists of *Eca*, (viz. the limiting points of the second order,

$$.1, \; .11, \; .111, \; \ldots),$$

and *Ecc*, which is the point 1, the single limiting point of the third order.

Thus *Ecc* is entirely contained in the second derived set, which, as remarked, is possible, although in the general case we can only assert that it is contained in the first derived set.

Similarly, by adding to the above set a finite number of limiting points of the first order of T_2, we get an example where *Eca* has some, but not all, of its points contained in E''.

If on the other hand, we also omit all the limiting points of the second order of T_2, we get an example where *Eca* is itself the third derived set, consisting of the point 1 alone.

On the same principle, using the set T_n, we can easily construct examples in which *Eca* is wholly, or in part, contained in any of the derived sets, up to the n^{th}.

If now *Ecc* be neither an isolated set nor dense in itself, we continue our process, and so on. Each successive coherence, and each successive adherence, will be contained in each successive derived set, so that we may speak generally of *the n^{th} derived coherence or adherence*, and the set E will consist of the n^{th} derived coherence, together with all the derived adherences, up to the $(n-1)^{th}$, and *Ea*.

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Continuing the process, either (1) we get to a stage at which the series of derived sets leaves off, (*i.e.* the first derived E' is countable), and therefore the series of adherences leaves off, and no coherence is left at the end, or else the latter series comes to an end before the former, leaving us either (2) with no coherence over, or (3) with one which is dense in itself, or, finally (4) all the derived sets exist, and corresponding to each we have a definite adherence.

In case (4) we examine whether or no we can "deduce" a set from the infinite series of coherences

$$Ec,\ Ecc,\ Eccc,\ \ldots\ \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Since these sets are not necessarily closed, they may have no common point, although the corresponding derived sets, being closed, certainly determine a deduced set; if this be the case the original set E consists only of the countably infinite series of adherences

$$Ea,\ Eca,\ Ecca,\ \ldots\ \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and is therefore countable, and has no component dense in itself.

Ex. 2. Take the set T_1 in the segment (0, 1), and insert in the intervals between its consecutive points from left to right, sets T_1, T_2, T_3, ... respectively. This is the stock example (Cantor's) of a countable set, whose first deduced set consists of a single point, the point 1.

From this set omit the point 1. The first deduced set still consists of the point 1 alone, but although, for every value of n there is an n^{th} derived adherence and coherence, we cannot deduce a set from the coherences, since they have no single point common.

If the deduced set of the coherences does exist, it consists entirely of limiting points of every preceding coherence, and therefore is contained in the first deduced set, just as the n^{th} derived coherence was in the n^{th} derived set. We may therefore properly designate it as *the first deduced coherence.* Any component of E which is dense in itself will be contained in it.

If the first deduced coherence does not exist, it does not follow that the set E has no component common with the first deduced set.

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Ex. 3. Take the closed set constructed at the beginning of Ex. 2, and omit all its limiting points except the point 1, which, as we saw, constitutes by itself the first deduced set. Ec is identical with Eca, and with the component common to E and its first deduced set, viz. it consists of the point 1 alone.

If the first deduced coherence exists, E consists of it, together with Ea and the derived adherences of every order; the former alone of these is not necessarily an isolated set, and, in particular, contains every component of E dense in itself. We can then analyse the first deduced coherence precisely as we did the set E, forming, on the one hand, a series of adherences and coherences, and, on the other, a series of derived sets, and continuing the process, if possible by deduction.

Proceeding in this way, either (1) the series of derived and deduced sets comes to an end, and therefore, the series of adherences comes to an end, and no coherence is left over, or else the latter series comes to an end, leaving us either, (2) with no coherence, or, (3) with one which is dense in itself, or, finally, (4) the series of derived and deduced sets does not come to an end, (E' being more than countable), but leads, after a countable series of steps, to the set E^*, while the series of adherences does not come to an end, and there is a definite coherence Ec^*, a component of E^*, left over at the final stage. In the following example Ec^* consists of a finite number of points.

Ex. 4. Let Y_1 denote the set consisting of the middle points of Cantor's ternary set,[1] together with a finite number of points of the latter set.

The set Y_1^* is then Cantor's set, and is the first derived set of Y_1.

The points common to Y_1 and Cantor's set constitute the first derived coherence of Y_1, and at the same time the set Y_1c^*.

Similarly in each of the following examples (5), Ec^* exists, and consists of the common points of E and E^*, the set Ec^* being, in the successive cases, more and more complicated in character.

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[1] That is, the end-points and external points of the following set of black intervals:—divide the segment (0, 1) into three equal parts, and blacken the middle part; repeat this process in the two segments left unblackened, and so on. See, for instance, W. H. Y., *Proc. Lond. Math. Society*, 35, p. 263.

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Ex. 5. Let Y_2 denote the set obtained by inserting in each of the black intervals of Cantor's ternary set a set similar to Y_1, and adding to these points a finite number of points of Cantor's set itself.

Here the first derived coherence is identical with Y_2c^*, which is the component common to Y_2 and Y_2^*.

Forming on this principle sets Y_3, Y_4, ... in succession, we see that for Y_n the first derived coherence is identical with Y_nc^*, which is the component common to Y_n and Y_n^*.

It is evident that, by using the sets Y_1, Y_2, ... as we did the sets T_1, T_2, ..., we can construct examples of a more and more complicated nature, illustrating the possibilities of the case; for instance, the following.

Ex. 6. In the largest black interval of Cantor's set introduce a set similar to Y_1; in the next two largest a set similar to Y_2, and so on; finally, take a finite number of points of Cantor's set itself. This set we call Y.

The set Y^* is reached at the first derivation, simultaneously with Yc^*, which consists of the common points of Y and Y^*.

These examples can be varied by taking instead of Y_1 in addition to a finite number of points of Cantor's set, a set having in each black interval of Cantor's set, a set T_1, or T_n, or any of the more complicated sets constructed by means of these. The set E^* is then not reached till the second derivation, or the n^{th}, or some more remote stage of proceedings.

Again it is clear that Ec^* may not exist, and yet E may have a component common with E^*, as the following example shows.

Ex. 7. Take one of the last examples in which E^* is not reached at the second derivation, and omit all the limiting points except the finite number of points belonging to Cantor's ternary set. Here the first derived coherence and adherence are identical with the common points of E and E^*, and the process of forming the successive adherences and coherences comes to an end at the second stage, while the process of derivation leads to new sets, and ultimately to E^*.

Similarly, by inserting suitably a sequence of points with its limiting point in one of the free intervals of the set last constructed, we can form an example in which some, but not all, of the points of the final adherence are points of the ultimate set, while Ec^* does not exist.

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If Ec^* exist, we can recommence our parallel processes with it as basis, analysing it into its adherents and coherents, and subjecting it to the processes of derivation and deduction. If neither set of processes lead us to a conclusion, we start afresh with Ec^{**}, and so on.

In this way we have before us a vista of adherences and coherences, extending possibly beyond the range of the distinct derived and deduced sets, as for instance in the examples (5) and (6). The derived and deduced sets can, we know, be arranged in countable order, we will now show that the same is true of the series of adherences in the most general case possible.

The ultimate coherence.[1]

§ 6. By Theorem 3 and the Corollary we know that a set is either countable, or has a Nucleus. In the latter case, by what has been pointed out, the Nucleus, being dense in itself, forms part of every possible coherence formed in the manner indicated, and those points of the given set which do not belong to the Nucleus are countable.

Choosing then one point from each adherence, we get a set of points which is at once countable, and in (1, 1)-correspondence with the adherences, thus *the adherences are themselves countable.*

Summing up, we have the following theorem :—

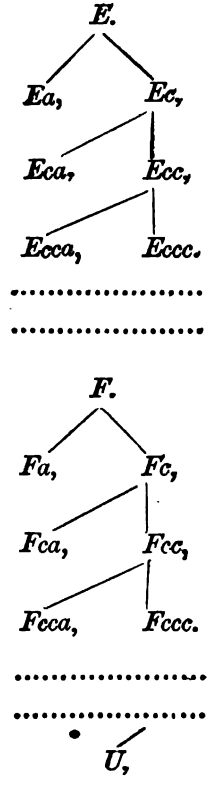
THEOREM 10. *Every set E which has no component dense in itself can be analysed into a countably infinite series of isolated sets (adherences); and every set E which has a component dense in itself into such a series, together with a single component U (the ultimate coherence), dense in itself and containing every component of E which is dense in itself.*

In particular, as already remarked, if E is more than countable, U will contain the Nucleus. If U does not coincide with the Nucleus, it can be shewn without difficulty to consist of two components, (1) the Nucleus, and (2) another component, dense in itself, no point of which is a point L'. If E be closed, the ultimate coherence, the set E^*, and the Nucleus are all identical. If E is not closed, the ultimate coherence is a component of E^*, but, as pointed out in the examples, does not necessarily contain every point common to E and E^*.

[1] Cantor uses the term "Total Inherence."

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This analysis of Cantor's is graphically illustrated in the form of a "tree."



where F denotes any coherence obtained from E, and U is the ultimate coherence.

It is of course allowable to start the tree at any point of ramification we like, for instance at F. Thus we see that any theorems which are true of Ea and Ec with respect to the subsequent adherences and coherences, or the derived and deduced sets of E, are true of Fa and Fc with respect to the adherences and coherences subsequent to them in the natural order, and the derived and deduced sets of E. As to the relations of Fa and Fc, and the derived and deduced sets of F, to the adherences and coherences *preceding* F in the natural order, we have pointed out in the examples that the possibilities are of the most varied nature.

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Thus, if G denote any derived or deduced set of E, and F the coherence corresponding to it, we have the following relations:

(1) F is contained in G;

(2) F' contains Fc, but not Fa, or any point of any adherence preceding F in the natural order;

(3) G' contains F', and therefore Fc, but may contain some or all points of Fa, or of any adherence preceding F in the natural order except Ea.

The above statements are then still true, if, instead of E, we start with any coherence preceding F in the natural order.

§ 8. We conclude this account with the following property of the adherences with respect to one another, a property to which the author was led in the course of work involving the application of the theory of coherences and adherences to that of the internal points of a sequence of sets of intervals.

THEOREM 11. *Each adherence consists entirely of points which are limiting points of every adherence preceding it in the natural order.*

That this is true for the derived adherences is easily seen, for Eca, being a component of Ec, consists of limiting points of E (that is, of Ea together with Ec), but no point of Eca is a limiting point of Ec; hence every point of Eca must be a limiting point of Ea. Similarly each derived adherence consists of limiting points of the adherence immediately preceding it in the natural order, and therefore, by induction, (since limiting points of limiting points are limiting points), of every adherence preceding it in the natural order.

That it is true of the first deduced adherence we may prove as follows.

Let F denote the first deduced coherence. If P be a point of F, but not a limiting point of Ea, it must be a limiting point of Ec; and, not being a limiting point of Ea, it cannot be a limiting point of Eca by the above, thus P must be a limiting point of Ecc, and so on. Thus P, being a limiting point of every one of the derived coherences, is a limiting point of F, and therefore a point of Fc and not of Fa. Similarly, if we had started with any derived adherence instead of E, as that of which F was not a limiting point, it would follow that F could not be a point of Fa. Thus every point of the first deduced adherence Fa is a limiting point of every derived adherence.

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The above method of proof is perfectly general, and may be extended by induction, to any adherence whatever, which proves the theorem.

Cor. The points of the second derived adherence are limiting points of the second order of Ea. Generally the points of any adherence are limiting points of corresponding order of Ea.

With respect to this corollary the remarks of the preceding paragraph apply, and we may substitute instead of *Ea* any adherence whatever, the other adherences in question being those subsequent to it in the natural order.

PERIODIC SOLUTIONS OF THE PROBLEM OF FOUR BODIES.

By Edgar Odell Lovett, Princeton, New Jersey.

In the problem of three bodies we owe to G. W. Hill the introduction of the fertile notion of periodic solutions; to Poincaré the perfecting of their mathematical theory; to Darwin a splendid collection of examples of such orbits.

In the problem of four bodies few examples of periodic orbits seem to have been given. In 1891 Lehmann-Filhés (*Astronomische Nachrichten*, No. 3033) found the exact solutions of the problem of four bodies corresponding to those of Lagrange in that of three; the collinear solutions of Lehmann-Filhés are periodic. In 1900 (*Transactions of the American Mathematical Society*, Vol. I., pp. 17–29) Moulton determined the exact solutions of the particular problem of four bodies where three are finite, in motion according to one or the other of the Lagrangian solutions, and the fourth infinitesimal. The above seem to be the only constructions published of periodic solutions of the problem of four bodies. To be sure, extensions of Lagrange's solutions had been given previous to the paper of Lehmann-Filhés, *e.g.* those of Veltmann (1875), Sloudsky (1878), and Hoppe (1879); Veltmann's has been subjected to criticism, and the generalisations of Sloudsky and Hoppe are more special than those of Lehmann-Filhés. See Whittaker's Report on the problem of three bodies, *British Association Report*, 1899.

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The following note is a modest contribution to this subject. It is one of the by-products of a course in celestial mechanics given at Princeton University during the last academic year. It suggests several questions for later discussion. It was itself suggested while reading a paper of Charlier's, cited further on, on periodic solutions of the problem of three bodies in which he reconstructs analytically certain of Darwin's orbits.

The methods employed are familiar. From mechanics we borrow the theory of the motion of a particle in a rotating field under forces possessing a potential, and from analysis, that of the periodic solutions of systems of linear differential equations with constant coefficients.

The problem is—given any three finite bodies in motion according to one or the other of the Lagrangian solutions of the problem of three bodies, and a fourth infinitesimal body acted on by the finite bodies according to Newton's law, but itself incapable of affecting their motion; locate the centres of libration, determine the roots of the characteristic equation, and construct periodic orbits in the immediate vicinity of those centres of libration about which periodic orbits may exist.

This problem is more general in character, but more limited in application than the simpler one of the restricted problem of three bodies. The nearest approach to a practical application of the former appears in a subsequent paragraph; the latter problem is well known to have found its justification in Hill's celebrated lunar theory, while Gyldén and Moulton have used it to give a meteoric explanation of the Gegenschein.

The case where the finite bodies move in the equilateral triangle solution of Lagrange seems unique in the symmetry maintained in the solution of the above questions.

The force-function appears in a symmetrical form expressed in terms of the three masses and their distances from the fourth particle; this is analogous to Darwin's form in the three body problem; when the fourth body is in the plane of the three, its form is a more obvious extension of that of Darwin.

As to centres of libration, the analytical conditions determining their positions possess solutions both without and within the plane of the finite bodies. Extraordinary mechanical conditions are necessary to the existence of those centres of libration situated outside of the plane of the finite bodies; in fact, in the numerical illustration of three unit masses given, the existence of such a centre is accounted for if one of the finite bodies is capable of simultaneously attracting finite bodies and repelling infinitesimal ones, that is, possessed of

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properties analogous to those attributed to the sun of our system to explain cometary phenomena while the comet is passing perihelion.

Periodic orbits exist about these remarkable centres.

A second set of conditions determines the centres of libration in the plane of the finite bodies.

Numerical application by way of illustration is made to the case of three unit masses. Periodic orbits are constructed in the immediate neighbourhood of the above anomalous centres in space, and also in the vicinity of those centres in the plane about which such orbits are possible; for the construction of the latter the formulæ of Charlier's paper referred to previously are immediately available on suitably modifying Darwin's form of the force-function.

In conclusion certain generalisations of these problems in process of solution are formulated. A postscript presents one of these, namely that of four bodies of arbitrary masses, or in fact of n bodies, on a straight line, analogous to the collinear solutions of Lagrange, studied under a method which Oppenheim used in the problem of three bodies, in a memoir inserted in the fourth volume of the publications of the *von Kuffner Observatory*. Relative to another of the questions enumerated, it may perhaps be remarked with propriety that, in the meantime the author has found that a method employed by Levi-Civita in the restricted problem of three bodies, may be applied effectively in the discussion of conditions for collisions in the problems of four bodies considered in the first part of this note.

1. Consider three finite bodies P_1, P_2, P_3 of masses m_1, m_2, m_3 respectively, in motion according to one of the Lagrangian equilateral triangle solutions of the problem of three bodies. Let the order of magnitude of the masses be

$$m_1 \geq m_2 \geq m_3, \tag{1}$$

and take

$$P_1P_2 = P_2P_3 = P_3P_1 = 1. \tag{2}$$

Let x, y, z be the coordinates of an infinitesimal body Q, referred to a system of rectangular axes whose origin is at P, the x-axis coinciding with P_1P_2, and the y-axis lying in the plane $P_1P_2P_3$.

The equations of motion of a particle (X, Y, Z) referred to rectangular axes rotating with angular velocity ω about the Z-axis, under the influence of forces whose potential is U,

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are

$$(3)\qquad \begin{cases} \dfrac{d}{dt}\left(\dfrac{dX}{dt}-\omega Y\right)-\omega\left(\dfrac{dY}{dt}+\omega X\right)=\dfrac{\partial U}{\partial X}, \\ \dfrac{d}{dt}\left(\dfrac{dY}{dt}+\omega X\right)+\omega\left(\dfrac{dX}{dt}-\omega Y\right)=\dfrac{\partial U}{\partial Y}, \\ \dfrac{d}{dt}\dfrac{dZ}{dt}=\dfrac{\partial U}{\partial Z}. \end{cases}$$

If in the present problem the origin be taken at the centre of inertia of the rotating system $P_1P_2P_3$, with the X-axis parallel to P_1P_2, and the Y-axis in the plane $P_1P_2P_3$, the coordinates of the infinitesimal body Q become

$$(4)\quad X=x-\frac{2m_2+m_3}{2(m_1+m_2+m_3)},\quad Y=y-\frac{m_3\sqrt{3}}{2(m_1+m_2+m_3)},\quad Z=z;$$

further

$$(5)\qquad \omega^2=m_1+m_2+m_3;$$

hence the equations of motion of the particle Q become

$$(6)\qquad \begin{cases} \dfrac{d^2x}{dt^2}-2\omega\dfrac{dy}{dt}=\omega^2x-\dfrac{2m_2+m_3}{2}+\dfrac{\partial U}{\partial x}, \\ \dfrac{d^2y}{dt^2}+2\omega\dfrac{dx}{dt}=\omega^2y-\dfrac{\sqrt{3}}{2}m_3+\dfrac{\partial U}{\partial y}, \\ \dfrac{d^2z}{dt^2}=\dfrac{\partial U}{\partial z}. \end{cases}$$

Now let ρ_1, ρ_2, ρ_3 be the tripolar coordinates of the point Q referred to P_1, P_2, P_3 at poles; then

$$(7)\qquad \begin{cases} \rho_1^2=x^2+y^2+z^2, \\ \rho_2^2=(x-1)^2+y^2+z^2, \\ \rho_3^2=(x-\frac{1}{2})^2+\left(y-\dfrac{\sqrt{3}}{2}\right)^2+z^2; \end{cases}$$

and if we put

$$(8)\quad \Omega=\tfrac{1}{2}\omega^2(x^2+y^2)-\tfrac{1}{2}(2m_2+m_3)x-\frac{\sqrt{3}}{2}m_3y+\frac{m_1}{\rho_1}+\frac{m_2}{\rho_2}+\frac{m_3}{\rho_3},$$

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the equations of motion (6) can be written

$$(9)\qquad \begin{cases} \dfrac{d^2x}{dt^2} - 2\omega\dfrac{dy}{dt} = \dfrac{\partial\Omega}{\partial x}, \\ \dfrac{d^2y}{dt^2} + 2\omega\dfrac{dx}{dt} = \dfrac{\partial\Omega}{\partial y}, \\ \dfrac{d^2z}{dt^2} = \dfrac{\partial\Omega}{\partial z}. \end{cases}$$

Observing that

$$(10)\qquad x = \frac{1+\rho_1^2-\rho_2^2}{2}, \quad y = \frac{1+\rho_1^2+\rho_2^2-2\rho_3^2}{2\sqrt{3}},$$

we find, after an easy reduction, that the function Ω can be written in the form

$$(11)\quad 6\Omega = (m_1+m_2+m_3)\{\rho_1^2(\rho_1^2-\rho_2^2-1)+\rho_2^2(\rho_2^2-\rho_3^2-1) + \rho_3^2(\rho_3^2-\rho_1^2-1)\} + \sum_1^3 i\, m_i\left(\rho_i^2+\frac{2}{\rho_i}\right) + \text{constant}.$$

This corresponds to Darwin's symmetrical form

$$(12)\qquad 2\Omega = m_1\left(\rho_1^2+\frac{2}{\rho_1}\right) + m_2\left(\rho_2^2+\frac{2}{\rho_2}\right)$$

of the force-function for the restricted problem of three bodies in the plane.

2. The case when the infinitesimal body lies in the plane of the finite bodies furnishes an interesting verification of the form (11). In fact the equations of motion for this case are

$$(13)\qquad \begin{cases} \dfrac{d^2x}{dt^2} - 2\omega\dfrac{dy}{dt} = \omega^2x - \dfrac{2m_1+m_2}{2} + \dfrac{\partial U'}{\partial x} \equiv \dfrac{\partial\Omega'}{\partial x}, \\ \dfrac{d^2y}{dt^2} + 2\omega\dfrac{dx}{dt} = \omega^2y - \dfrac{\sqrt{3}}{2}m_2 + \dfrac{\partial U'}{\partial y} \equiv \dfrac{\partial\Omega'}{\partial y}, \end{cases}$$

where

$$(14)\quad \Omega' = \tfrac{1}{2}\omega^2(x^2+y^2) - \frac{2m_1+m_2}{2}x - \frac{\sqrt{3}}{2}m_2y + \frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3};$$

and in virtue of the relations

$$(15)\quad r_1^2 = x^2+y^2,\ r_2^2 = (x-1)^2+y^2,\ r_3^2 = (x-\tfrac{1}{2})^2 + \left(y-\frac{\sqrt{3}}{2}\right)^2,$$

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we have

$$(16)\quad 2\Omega' = m_1\left(r_1^2+\frac{2}{r_1}\right)+m_2\left(r_2^2+\frac{2}{r_2}\right)+m_3\left(r_3^2+\frac{2}{r_3}\right)+\text{constant}.$$

But the r's are not all independent; the side of the equilateral triangle being unity, we find the relation

$$(17)\quad r_1^2(r_1^2-r_2^2-1)+r_2^2(r_2^2-r_3^2-1)+r_3^2(r_3^2-r_1^2-1)+1=0,$$

and the connection between the force-functions for the plane and space problems is apparent.

3. Returning now to the original problem in space, let (a, b, c) be a centre of libration and ξ, η, ζ the coordinates of the infinitesimal body referred to it as origin; then

$$(18)\qquad x=a+\xi,\ y=b+\eta,\ z=c+\zeta.$$

From the equations of motion we have

$$(19)\quad \begin{cases} \dfrac{d^2\xi}{dt^2}-2\omega\dfrac{d\eta}{dt}=\dfrac{\partial\Omega}{\partial a}+\xi\dfrac{\partial^2\Omega}{\partial a^2}+\eta\dfrac{\partial^2\Omega}{\partial a\partial b}+\zeta\dfrac{\partial^2\Omega}{\partial a\partial c}+\tfrac{1}{2}\left\{\xi^2\dfrac{\partial^3\Omega}{\partial a^3}+\ldots\right\}+\ldots, \\ \dfrac{d^2\eta}{dt^2}+2\omega\dfrac{d\xi}{dt}=\dfrac{\partial\Omega}{\partial b}+\xi\dfrac{\partial^2\Omega}{\partial a\partial b}+\eta\dfrac{\partial^2\Omega}{\partial b^2}+\zeta\dfrac{\partial^2\Omega}{\partial b\partial c}+\tfrac{1}{2}\left\{\xi^2\dfrac{\partial^3\Omega}{\partial a^2\partial b}+\ldots\right\}+\ldots, \\ \dfrac{d^2\zeta}{dt^2}=\dfrac{\partial\Omega}{\partial c}+\xi\dfrac{\partial^2\Omega}{\partial a\partial c}+\eta\dfrac{\partial^2\Omega}{\partial b\partial c}+\zeta\dfrac{\partial^2\Omega}{\partial c^2}+\tfrac{1}{2}\left\{\xi^2\dfrac{\partial^3\Omega}{\partial a^2\partial c}+\ldots\right\}+\ldots \end{cases}$$

At the centre of libration we must have

$$(20)\quad \frac{\partial\Omega}{\partial a}=0,\quad \frac{\partial\Omega}{\partial b}=0,\quad \frac{\partial\Omega}{\partial c}=0;$$

and if attention be restricted to such small orbits as are situated in the immediate vicinity of (a, b, c) the powers of ξ, η, ζ higher than the first in the developments of $\frac{\partial\Omega}{\partial x}$, $\frac{\partial\Omega}{\partial y}$, $\frac{\partial\Omega}{\partial z}$ may be neglected; accordingly the differential equations of these orbits become

$$(21)\quad \begin{cases} \dfrac{d^2\xi}{dt^2}-2\omega\dfrac{d\eta}{dt}=\xi\Omega_{aa}+\eta\Omega_{ab}+\zeta\Omega_{ac}, \\ \dfrac{d^2\eta}{dt^2}+2\omega\dfrac{d\xi}{dt}=\xi\Omega_{ab}+\eta\Omega_{bb}+\zeta\Omega_{bc}, \\ \dfrac{d^2\zeta}{dt^2}=\xi\Omega_{ac}+\eta\Omega_{bc}+\zeta\Omega_{cc}. \end{cases}$$

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The equations (20) are to furnish the positions of the centres of libration. To evaluate them we have the following relations

$$(22)\qquad \rho_i^2 = (a - x_i)^2 + (b - y_i)^2 + c^2, \quad i = 1, 2, 3;$$

$$(23)\qquad \sum_1^3 i\, m_i x_i = 0, \quad \sum_1^3 i\, m_i y_i = 0, \quad y_1 - y_2 = 0;$$

$$(24)\qquad (x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_2 - x_3)^2 + (y_2 - y_3)^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2 = 1.$$

The equations (20) can be written

$$(25)\qquad \begin{cases} \dfrac{\partial \Omega}{\partial a} \equiv \dfrac{\partial \Omega}{\partial \rho_1}\dfrac{a - x_1}{\rho_1} + \dfrac{\partial \Omega}{\partial \rho_2}\dfrac{a - x_2}{\rho_2} + \dfrac{\partial \Omega}{\partial \rho_3}\dfrac{a - x_3}{\rho_3} = 0, \\ \dfrac{\partial \Omega}{\partial b} \equiv \dfrac{\partial \Omega}{\partial \rho_1}\dfrac{b - y_1}{\rho_1} + \dfrac{\partial \Omega}{\partial \rho_2}\dfrac{b - y_2}{\rho_2} + \dfrac{\partial \Omega}{\partial \rho_3}\dfrac{b - y_3}{\rho_3} = 0, \\ \dfrac{\partial \Omega}{\partial c} \equiv \dfrac{\partial \Omega}{\partial \rho_1}\dfrac{c}{\rho_1} + \dfrac{\partial \Omega}{\partial \rho_2}\dfrac{c}{\rho_2} + \dfrac{\partial \Omega}{\partial \rho_3}\dfrac{c}{\rho_3} = 0; \end{cases}$$

these equations (25) demand either 1°

$$(26)\qquad \begin{vmatrix} \dfrac{a - x_1}{\rho_1}, & \dfrac{a - x_2}{\rho_2}, & \dfrac{a - x_3}{\rho_3} \\ \dfrac{b - y_1}{\rho_1}, & \dfrac{b - y_2}{\rho_2}, & \dfrac{b - y_3}{\rho_3} \\ \dfrac{c}{\rho_1}, & \dfrac{c}{\rho_2}, & \dfrac{c}{\rho_3} \end{vmatrix} = 0,$$

or 2°

$$(27)\qquad \frac{\partial \Omega}{\partial \rho_1} = 0, \quad \frac{\partial \Omega}{\partial \rho_2} = 0, \quad \frac{\partial \Omega}{\partial \rho_3} = 0.$$

The first condition (26) is equivalent to

$$(28)\qquad \frac{c}{\rho_1 \rho_2 \rho_3} \begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix} = 0;$$

hence, since neither can any ρ be infinite nor the area of the equilateral triangle be zero, we must have

$$(29)\qquad c = 0;$$

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that is, the infinitesimal body must lie in the plane of the three finite bodies. To this case we shall return later.

The second condition (27) gives

$$(30)\begin{cases}\dfrac{\partial\Omega}{\partial\rho_1}\equiv\dfrac{2}{\rho_1^3}\{\rho_1^3(2\rho_1^2-\rho_2^2-\rho_3^2-1)(m_1+m_2+m_3)+3(\rho_1^3-1)m_1\}=0,\\[2ex] \dfrac{\partial\Omega}{\partial\rho_2}\equiv\dfrac{2}{\rho_2^3}\{\rho_2^3(2\rho_2^2-\rho_3^2-\rho_1^2-1)(m_1+m_2+m_3)+3(\rho_2^3-1)m_2\}=0,\\[2ex] \dfrac{\partial\Omega}{\partial\rho_3}\equiv\dfrac{2}{\rho_3^3}\{\rho_3^3(2\rho_3^2+\rho_1^2-\rho_2^2-1)(m_1+m_2+m_3)+3(\rho_3^3-1)m_3\}=0.\end{cases}$$

It may be remarked in passing that any two of these equations can be replaced by any two of the three following equations:

$$(31)\begin{cases} m_1\left(1-\dfrac{1}{\rho_1^3}\right)-m_2\left(1-\dfrac{1}{\rho_2^3}\right)+(m_1+m_2+m_3)(\rho_1^2-\rho_2^2)=0,\\[2ex] m_2\left(1-\dfrac{1}{\rho_2^3}\right)-m_3\left(1-\dfrac{1}{\rho_3^3}\right)+(m_1+m_2+m_3)(\rho_2^2-\rho_3^2)=0,\\[2ex] m_3\left(1-\dfrac{1}{\rho_3^3}\right)-m_1\left(1-\dfrac{1}{\rho_1^3}\right)+(m_1+m_2+m_3)(\rho_3^2-\rho_1^2)=0.\end{cases}$$

It may also be observed that all the ρ's cannot be equal as is possible in the corresponding case for the restricted problem of three bodies in the plane; for, letting

$$(32)\qquad \rho_1=\rho_2=\rho_3=R,$$

the equations (30) demand either that

$$(33)\qquad m_1=m_2=m_3=0,$$

or that

$$(34)\qquad m_1+m_2+m_3=0,\quad R_3=1.$$

The above equations (30) of condition are three in number, each linearly homogeneous in the three quantities m_1, m_2, m_3; hence we must have

$$(35)\qquad D\equiv\begin{vmatrix}\lambda+\alpha, & \lambda, & \lambda\\ \mu, & \mu+\beta, & \mu\\ \nu, & \nu, & \nu+\gamma\end{vmatrix}=0,$$

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where

$$(36)\quad \begin{cases} \lambda = \rho_1^3(2\rho_1^2 - \rho_2^2 - \rho_3^2 - 1), & \alpha = 3(\rho_1^3 - 1), \\ \mu = \rho_2^3(2\rho_2^2 - \rho_3^2 - \rho_1^2 - 1), & \beta = 3(\rho_2^3 - 1), \\ \nu = \rho_3^3(2\rho_3^2 - \rho_1^2 - \rho_2^2 - 1), & \gamma = 3(\rho_3^3 - 1), \end{cases}$$

or in expanded form

$$\begin{aligned}(37)\qquad & (\rho_1^3 - 1)\{(\rho_2^3 - \rho_3^3)(\rho_2^2 - \rho_3^2) + \rho_3^3 - 1\} \\ & + (\rho_2^3 - 1)\{(\rho_3^3 - \rho_1^3)(\rho_3^2 - \rho_1^2) + \rho_1^3 - 1\} \\ & + (\rho_3^3 - 1)\{(\rho_1^3 - \rho_2^3)(\rho_1^2 - \rho_2^2) + \rho_2^3 - 1\} = 0.\end{aligned}$$

4. Let us consider the case of three equal finite masses. The determinant D is identically zero, and the preceding equations of condition become

$$(38)\qquad X^3Y^3 + Y^3Z^3 + Z^3X^3 = 0,$$

$$(39)\qquad 3X^3Y^3 - 3X^5Y^3 - X^3 + Y^3 = 0,$$

$$(40)\qquad 3X^3Z^3 - 3X^5Z^3 - X^3 + Z^3 = 0,$$

where for convenience ρ_1, ρ_2, ρ_3 are replaced respectively by X, Y, Z. The high degree of these equations seemed to promise the first approximation to a star-cluster, but the expectations were not realized. The real solutions can be more readily found by interpreting X, Y, Z as rectangular point-coordinates in an ordinary three-dimensional space. We have then to determine the points where the cone (38) is cut by the curve of intersection of the congruent right cylinders (39) and (40).

The equations (39) and (40) may be written

$$(41)\quad (X - Y)\{3X^3Y^3(X + Y) + X^3 + XY + Y^3\} = 0,$$

$$(42)\quad (Z - X)\{3Z^3X^3(Z + X) + Z^3 + ZX + X^3\} = 0;$$

the right section of the congruent cylinders may be readily constructed:—for example, the section of the first one by the XY-plane consists in the first quadrant of the line $X - Y = 0$; in the second, of an infinite branch asymptotic to the Y-axis and to the line $X + Y = 0$; in the third quadrant, of the line $X - Y = 0$, and of an infinite branch asymptotic to the X-axis and the Y-axis; in the fourth quadrant, of an infinite branch asymptotic to the X-axis and the line $X + Y = 0$.

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The two cylinders

$$3X^2Y^2(X+Y)+X^3+XY+Y^2=0, \tag{43}$$

$$3Z^2X^2(Z+X)+Z^3+ZX+X^2=0, \tag{44}$$

meet only in the plane

$$Y-Z=0. \tag{45}$$

The plane (45) cuts the cone (38) in the straight line

$$-2^{\frac{1}{3}}X=Y=Z; \tag{46}$$

this line meets the intersection of (43) and (44) in the real point

$$Y=Z=\sqrt[5]{\left(\frac{2^{\frac{2}{3}}}{2^{\frac{2}{3}}-1}\right)},\quad X=-\frac{Y}{\sqrt[5]{2}}. \tag{47}$$

The plane

$$X-Y=0 \tag{48}$$

cuts (38) in the line

$$-2^{\frac{1}{3}}Z=X=Y; \tag{49}$$

this line cuts the cylinder (44) in the point

$$X=Y=\sqrt[5]{\left(\frac{2^{\frac{2}{3}}}{2^{\frac{2}{3}}-1}\right)},\quad Z=-\frac{X}{\sqrt[5]{2}}. \tag{50}$$

The plane

$$Z-X=0 \tag{51}$$

intersects (38) in the element

$$-2^{\frac{1}{3}}Y=Z=X; \tag{52}$$

the line intersects the cylinder (43) in the point

$$Z=X=\sqrt[5]{\left(\frac{2^{\frac{2}{3}}}{2^{\frac{2}{3}}-1}\right)},\quad Y=-\frac{Z}{\sqrt[5]{2}}. \tag{53}$$

The origin is obviously a point common to the three surfaces, but it is excluded since no ρ can be zero consistently with the mechanical conditions of the problem. The above three points are the only other real points common to the three surfaces.

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Evaluating the radicals we have the following approximate values of those ρ's which are capable of determining centres of libration:

$$(54)\quad \begin{cases} \rho_1=\rho_2=k,\ \rho_3=l;\ \rho_1=l,\ \rho_2=\rho_3=k;\ \rho_3=\rho_1=k,\ \rho_2=l; \\ k=1.21997,\ l=-0.96829. \end{cases}$$

These values substituted in the equations (25) give the following centres of libration, the common value of the masses being unity:

$$(55)\quad \begin{cases} a=0, \\ b=0.31797, \\ c=\pm 0.93290; \end{cases} \quad \begin{cases} a=-0.27537, \\ b=-0.15898, \\ c=\pm 0.93290; \end{cases} \quad \begin{cases} a=+0\ 27537, \\ b=-0.15898, \\ c=\pm 0.93290. \end{cases}$$

These points lie in planes parallel to the plane of the three finite bodies and equidistant from it, forming two equilateral triangles of about half the linear dimensions of the original triangle; their projections lie on the axes of symmetry of the original triangle at equal distances from its centre of mass.

We remark the unusual physical conditions which the existence of these centres demands in order to interpret mechanically the negative ρ (geometrically the sign of a tripolar coordinate in ordinary space seems incapable of an unambiguous interpretation). A centre such as the above may exist if one of the finite bodies is capable of simultaneously attracting finite bodies and repelling infinitesimal ones; that is, possessed of properties analogous to those attributed to the sun of our system to account for cometary phenomena when the comet passes perihelion.

It would perhaps be more satisfactory to retain only positive values of the ρ's in the discussion and take the function U in the form

$$-\frac{m_1}{\rho_1}+\frac{m_2}{\rho_2}+\frac{m_3}{\rho_3},$$

thus allowing the mass m_1 to repel the infinitesimal particle. The function Ω is then given by

$$6\omega=\{\rho_1^2(\rho_1^2-\rho_2^2-1)+\rho_2^2(\rho_2^2-\rho_3^2-1)+\rho_3^2(\rho_3^2-\rho_1^2-1)\}\sum_1^3 m_i$$

$$+m_1\left(\rho_1^2-\frac{2}{\rho_1}\right)+m_2\left(\rho_2^2+\frac{2}{\rho_2}\right)+m_3\left(\rho_3^2+\frac{2}{\rho_3}\right)+\text{constant}.$$

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The conditions (27) remain the same as written above with the exception of the first, which becomes

$$\frac{\partial \omega}{\partial \rho_1} \equiv \frac{2}{\rho_1^3}\{\rho_1^2(2\rho_1^2 - \rho_2^2 - \rho_3^2 - 1)\sum_1^3 m_i + 3(\rho_1^2 + 1)m_1\} = 0.$$

Further for the case of three equal masses the equations (38), (39), and (40) are replaced respectively by

$$x^2y^2 - y^2z^2 + z^2x^2 = 0,$$
$$3x^2y^2(x^2 - y^2) + x^2 + y^2 = 0,$$
$$3x^2z^2(x^2 - z^2) + x^2 + z^2 = 0;$$

the following point of these three surfaces

$$y = z = \sqrt[5]{\left(\frac{2^{\frac{3}{2}}}{2^{\frac{3}{2}} - 1}\right)} = \rho_2 = \rho_3 = 1.21997, \quad x = \frac{y}{\sqrt[3]{2}} = \rho_1 = 0.96829,$$

is sufficient for our purposes (in fact there are negative numbers among the coordinates of the other points of intersection, excepting the origin, and ought to be excluded as above).

The equations (22) for the coordinates of the centre of libration as functions of the ρ's give

$$a = \frac{\rho_1^2 - \rho_2^2}{2}, \quad b = \frac{\rho_1^2 - \rho_2^2}{2\sqrt{3}}, \quad c^2 = \rho_2^2 + \tfrac{1}{3}\{(\rho_1^2 - \rho_2^2) - (\rho_1^2 - \rho_2^2)^2 - 1\};$$

introducing the numerical values of the ρ's we find

$$a = -0.27537, \quad b = -0.15898, \quad c = \pm 0.93290,$$

which are identical with those found previously. This of course finds its explanation in the fact that only the squares of the ρ's occur in the equations (22).

5. The differential equations (21) of the orbits of the infinitesimal body in the immediate neighbourhood of a centre of libration constitute a system of linear differential equations with constant coefficients; they possess then a solution of the form

$$(56) \qquad \xi = Ae^{\lambda t}, \quad \eta = Be^{\lambda t}, \quad \eta = Ce^{\lambda t},$$

where A, B, C are determined by the equations

$$(57) \quad \begin{cases} (\lambda^2 - \Omega_{aa})A - (2\omega\lambda + \Omega_{ab})B - \Omega_{ac}C = 0, \\ (2\omega\lambda - \Omega_{ab})A + (\lambda^2 - \Omega_{bb})B - \Omega_{bc}C = 0, \\ \Omega_{ca}A + \Omega_{bc}B - (\lambda^2 - \Omega_{cc})C = 0; \end{cases}$$

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this algebraic system being linear and homogeneous in A, B, C, we must have

$$(58)\quad \begin{vmatrix} \lambda^2-\Omega_{aa}\ , & -2\omega\lambda-\Omega_{ab}, & \Omega_{ac} \\ 2\omega\lambda-\Omega_{ab}, & \lambda^2-\Omega_{bb}\ , & \Omega_{bc} \\ \Omega_{ca}\ , & \Omega_{bc}\ , & \lambda^2-\Omega_{cc} \end{vmatrix} = 0;$$

or, in expanded form in powers of λ,

$$(59)\quad \begin{cases} \lambda^6+(4\omega^2-\Omega_{aa}-\Omega_{bb}-\Omega_{cc})\lambda^4 \\ \quad +(\Omega_{aa}\Omega_{bb}+\Omega_{bb}\Omega_{cc}+\Omega_{cc}\Omega_{aa}-\Omega^2_{ab}-\Omega^2_{bc}-\Omega^2_{ca}-4\omega^2\Omega_{cc})\lambda^2 \\ \quad +\Omega_{aa}\Omega^2_{bc}+\Omega_{bb}\Omega^2_{ca}+\Omega_{cc}\Omega^2_{ab}-\Omega_{aa}\Omega_{bb}\Omega_{cc}-2\Omega_{ab}\Omega_{bc}\Omega_{ca}=0. \end{cases}$$

This is a cubic in λ^2. The character of the orbits in the immediate vicinity of a centre of libration depends on the nature of the roots of this cubic. For a periodic solution it is necessary that one of the roots be negative.

The coefficients of this characteristic equation are functions of the second derivatives of the function Ω. We have

$$(60)\quad \begin{cases} \Omega_{aa}=\sum_1^3 i\ m_i\left\{\frac{3}{\rho_i^5}(a-x_i)^2-\frac{1}{\rho_i^3}\right\}+\phi(a,x), \\ \Omega_{bb}=\sum_1^3 i\ m_i\left\{\frac{3}{\rho_i^5}(b-y_i)^2-\frac{1}{\rho_i^3}\right\}+\phi(b,y), \\ \Omega_{cc}=\sum_1^3 i\ m_i\left\{\frac{3c^2}{\rho_1^5}-\frac{1}{\rho_i^3}\right\}, \\ \Omega_{ab}=r_{123}+r_{231}+r_{312}, \\ \Omega_{bc}=3c\sum_1^3 i\ \frac{m_i}{\rho_i^5}(b-y_i), \\ \Omega_{ca}=3c\sum_1^3 i\ \frac{m_i}{\rho_i^5}(a-x_i), \\ \phi(k,l)=\frac{4}{3}(m_1+m_2+m_3)\{(k-l_1)^2+(k-l_2)^2+(k-l_3)^2 \\ \qquad -(k-l_1)(k-l_2)-(k-l_2)(k-l_3)-(k-l_3)(k-l_1)\}, \\ r_{ijk}=\frac{a-x_i}{3}\left\{\frac{9m_i}{\rho_i^5}(b-y_i)\right. \\ \qquad \left.+2(m_1+m_2+m_3)[2(b-y_i)-(b-y_j)-(b-y_k)]\right\}. \end{cases}$$

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The symmetry of the results already found show that it will be sufficient to study the orbits in the neighbourhood of any one of the six centres. Take, for example, the centre

$$a = 0, \quad b = 0.31797, \quad c = 0.93290, \tag{61}$$

at which we have

$$\rho_1 = \rho_2 = 1.21997, \quad \rho_3 = -0.96829. \tag{62}$$

The formulæ (60) give for these data and masses equal to unity,

$$\begin{cases} \Omega_{aa} = 3.55506, & \Omega_{bb} = 3.58006, & \Omega_{cc} = -1.13\,509, \\ \Omega_{ab} = 0, & \Omega_{bc} = -2.10940, & \Omega_{ca} = 0. \end{cases} \tag{63}$$

Writing the equation (59) in the form

$$\lambda^6 + \alpha\lambda^4 + \beta\lambda^2 + \gamma = 0, \tag{64}$$

the values (63) give

$$\alpha = 5.99997, \quad \beta = 13.79983, \quad \gamma = 30.26514. \tag{65}$$

This equation has a negative root whose approximate value is

$$\lambda^2 = -4.42703. \tag{66}$$

6. To each value of λ the equations (57) give corresponding values of the ratios

$$A : B : C, \tag{67}$$

and the general integral of the differential equations (21) has the form

$$\xi = \sum_1^6 A_j e^{\lambda_j t}, \quad \eta = \sum_1^6 B_j e^{\lambda_j t}, \quad \zeta = \sum_1^6 C_j e^{\lambda_j t}, \tag{68}$$

the values of A_j, B_j, C_j being related through the equations

$$\begin{cases} \dfrac{A_j}{B_j} = \begin{vmatrix} -2\omega\lambda_j - \Omega_{ab}, & -\Omega_{ca} \\ \lambda_j^2 - \Omega_{bb}, & -\Omega_{bc} \end{vmatrix} \Big/ \begin{vmatrix} \lambda_j^2 - \Omega_{aa}, & \Omega_{ca} \\ 2\omega\lambda_j - \Omega_{ab}, & \Omega_{bc} \end{vmatrix}, \\ \dfrac{C_j}{B_j} = \begin{vmatrix} \lambda^2 - \Omega_{aa}, & -2\omega\lambda - \Omega_{ab} \\ 2\omega\lambda - \Omega_{ab}, & \lambda^2 - \Omega_{bb} \end{vmatrix} \Big/ \begin{vmatrix} \lambda^2 - \Omega_{aa}, & \Omega_{ca} \\ 2\omega\lambda - \Omega_{ab}, & \Omega_{bc} \end{vmatrix}. \end{cases} \tag{69}$$

Six of the coefficients A_j, B_j, C_j are arbitrary; regarding

$$B_1, B_2, B_3, B_4, B_5, B_6 \tag{70}$$

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as these six arbitraries, the above equations give the corresponding values of A_j and C_j as functions of $B_1, B_2, \ldots, B_6$.

Let λ_j, λ_2 be two conjugate pure imaginary roots of the equation in λ, and put

$$\nu^2 = -\lambda^2, \tag{71}$$

so that ν_1 and ν_2 are real.

We have then

$$\begin{cases} \dfrac{A_j}{B_j} = -\dfrac{\begin{vmatrix} \Omega_{ab}+2i\omega\nu_j, & \Omega_{aa} \\ \Omega_{bb}+\nu_j^2, & \Omega_{bc} \end{vmatrix}}{\begin{vmatrix} \Omega_{aa}+\nu_j^2, & \Omega_{ca} \\ \Omega_{ab}-2i\omega\nu_j, & \Omega_{bc} \end{vmatrix}}, \\ \dfrac{C_j}{B_j} = -\dfrac{\begin{vmatrix} \Omega_{aa}+\nu_j^2, & \Omega_{ab}+2i\omega\nu_j \\ \Omega_{ab}-2i\omega\nu_j, & \Omega_{bb}+\nu_j^2 \end{vmatrix}}{\begin{vmatrix} \Omega_{aa}+\nu_j^2, & \Omega_{ca} \\ \Omega_{ab}-2i\omega\nu_j, & \Omega_{bc} \end{vmatrix}}, \end{cases} \quad j=1, 2, \; i=\sqrt{(-1)}, \tag{72}$$

where

$$\nu_2 = -\nu_1. \tag{73}$$

Let us now separate $A_1, A_2, B_1, B_2, C_1, C_2$, each into its real and imaginary parts; since the orbits are to be real we shall have

$$\begin{cases} A_1 = \alpha_1 + i\alpha_2, & B_1 = \beta_1 + i\beta_2, & C_1 = \gamma_1 + i\gamma_2, \\ A_2 = \alpha_1 - i\alpha_2, & B_2 = \beta_1 - i\beta_2, & C_2 = \gamma_1 - i\gamma_2; \end{cases} \tag{74}$$

hence if the arbitrary constants be so chosen that

$$B_3 = B_4 = B_5 = B_6 = 0, \tag{75}$$

we have a periodic solution of the form

$$\begin{cases} \xi = A_1 e^{\nu_1 t} + A_2 e^{\nu_2 t} = 2\alpha_1 \cos \nu_2 t + 2\alpha_2 \sin \nu_2 t, \\ \eta = B_1 e^{\nu_1 t} + B_2 e^{\nu_2 t} = 2\beta_1 \cos \nu_2 t + 2\beta_2 \sin \nu_2 t, \\ \zeta = C_1 e^{\nu_1 t} + C_2 e^{\nu_2 t} = 2\gamma_1 \cos \nu_2 t + 2\gamma_2 \sin \nu_2 t. \end{cases} \tag{76}$$

The Cartesian equations may be obtained by eliminating the time by means of these expressions for ξ, η, ζ. Effecting this elimination, we have

$$\begin{cases} \begin{vmatrix} \xi, & \alpha_2 \\ \eta, & \beta_2 \end{vmatrix}^2 + \begin{vmatrix} \alpha_1, & \xi \\ \beta_1, & \eta \end{vmatrix}^2 = 4 \begin{vmatrix} \alpha_1, & \alpha_2 \\ \beta_1, & \beta_2 \end{vmatrix}^2, \\ \begin{vmatrix} \eta, & \beta_2 \\ \zeta, & \gamma_2 \end{vmatrix}^2 + \begin{vmatrix} \beta_1, & \eta \\ \gamma_1, & \zeta \end{vmatrix}^2 = 4 \begin{vmatrix} \beta_1, & \beta_2 \\ \gamma_1, & \gamma_2 \end{vmatrix}^2, \\ \begin{vmatrix} \zeta, & \gamma_2 \\ \xi, & \alpha_2 \end{vmatrix}^2 + \begin{vmatrix} \gamma_1, & \zeta \\ \alpha_1, & \xi \end{vmatrix}^2 = 4 \begin{vmatrix} \gamma_1, & \gamma_2 \\ \alpha_1, & \alpha_2 \end{vmatrix}^2. \end{cases} \tag{77}$$

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It remains now to eliminate the α's and γ's; to this end we substitute in equations (72) the values of A_2, B_2, C_2 from (74); on equating the real and imaginary parts of the equations

$$(78)\quad \begin{cases} \dfrac{\alpha_1 - i\alpha_2}{\beta_1 - i\beta_2} = -\left| \begin{matrix} \Omega_{ab}+2i\omega\nu_2, & \Omega_{ca} \\ \Omega_{bb}+\nu_2^2, & \Omega_{bc} \end{matrix} \right| \Big/\!\Big| \left| \begin{matrix} \Omega_{aa}+\nu_2^2, & \Omega_{ca} \\ \Omega_{ab}-2i\omega\nu_2, & \Omega_{bc} \end{matrix} \right|, \\ \dfrac{\gamma_1 - i\gamma_2}{\beta_1 - i\beta_2} = -\left| \begin{matrix} \Omega_{aa}+\nu_2^2, & \Omega_{ab}+2i\omega\nu_2 \\ \Omega_{ab}-2i\omega\nu_2, & \Omega_{bb}+\nu_2^2 \end{matrix} \right| \Big/\!\Big| \left| \begin{matrix} \Omega_{aa}+\nu_2^2, & \Omega_{ca} \\ \Omega_{ab}-2i\omega\nu_2, & \Omega_{bc} \end{matrix} \right|, \end{cases}$$

thus formed, we have the following equations for α_1, α_2, γ_1, γ_2,

$$(79)\quad \begin{cases} P\alpha_1 + Q\alpha_2 = R\beta_1 - S\beta_2, \\ Q\alpha_1 - P\alpha_2 = -S\beta_1 - R\beta_2, \\ P\gamma_1 + Q\gamma_2 = -T\beta_1, \\ Q\gamma_1 - P\gamma_2 = T\beta_2, \end{cases}$$

where

$$(80)\quad \begin{cases} P = (\Omega_{aa}+\nu_2^2)\,\Omega_{bc} - \Omega_{ab}\Omega_{ca}, \quad Q = 2\omega\nu_2\Omega_{ca}, \\ R = (\Omega_{bb}+\nu_2^2)\,\Omega_{ca} - \Omega_{ab}\Omega_{bc}, \quad S = 2\omega\nu_2\Omega_{bc}, \\ T = (\Omega_{aa}+\nu_2^2)\,(\Omega_{bb}+\nu_2^2) - \Omega_{ab}^2 - 4\omega^2\nu_2^2. \end{cases}$$

From (79) we deduce at once

$$(81)\quad \begin{cases} (P^2+Q^2)\,(\alpha_1^2+\alpha_2^2) = (R^2+S^2)\,(\beta_1^2+\beta_2^2), \\ (P^2+Q^2)\,(\gamma_1^2+\gamma_2^2) = T^2\,(\beta_1^2+\beta_2^2). \end{cases}$$

Further, in virtue of the formula

$$(82)\quad \frac{\rho+i\sigma}{\rho'+i\sigma'} = \frac{\rho\rho'+\sigma\sigma'}{\rho'^2+\sigma'^2} + i\frac{\rho'\sigma-\rho\sigma'}{\rho'^2+\sigma'^2},$$

we write

$$(83)\quad \frac{A_2}{B_2} = \frac{\alpha_1\beta_1+\alpha_2\beta_2}{\beta_1^2+\beta_2^2} + i\frac{\alpha_1\beta_2-\alpha_2\beta_1}{\beta_1^2+\beta_2^2}, \quad \frac{C_2}{B_2} = \frac{\gamma_1\beta_1+\gamma_2\beta_2}{\beta_1^2+\beta_2^2} + i\frac{\gamma_1\beta_2-\gamma_2\beta_1}{\beta_1^2+\beta_2^2};$$

also

$$(84)\quad \begin{cases} (P^2+Q^2)\,\dfrac{A_2}{B_2} = PR - QS - i\,(PS+QR), \\ (P^2+Q^2)\,\dfrac{C_2}{B_2} = -PT + iQT; \end{cases}$$

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whence, on comparing the values of these ratios, we have

$$(85)\quad \begin{cases} (P^2+Q^2)(\alpha_1\beta_1+\alpha_2\beta_2) = (PR-QS)(\beta_1^2+\beta_2^2), \\ (P^2+Q^2)(\alpha_1\beta_2-\alpha_2\beta_1) = -(PS+QR)(\beta_1^2+\beta_2^2), \\ (P^2+Q^2)(\beta_1\gamma_1+\beta_2\gamma_2) = -PT(\beta_1^2+\beta_2^2), \\ (P^2+Q^2)(\beta_1\gamma_2-\beta_2\gamma_1) = -QT(\beta_1^2+\beta_2^2). \end{cases}$$

Expanding the equations (77), we have

$$(86)\quad \begin{cases} (\beta_1^2+\beta_2^2)\xi^2+(\alpha_1^2+\alpha_2^2)\eta^2-2(\alpha_1\beta_1+\alpha_2\beta_2)\xi\eta=4(\alpha_1\beta_2-\alpha_2\beta_1)^2, \\ (\gamma_1^2+\gamma_2^2)\eta^2+(\beta_1^2+\beta_2^2)\zeta^2-2(\beta_1\gamma_1+\beta_2\gamma_2)\eta\zeta=4(\beta_1\gamma_2-\beta_2\gamma_1)^2, \\ (\alpha_1^2+\alpha_2^2)\zeta^2+(\gamma_1^2+\gamma_2^2)\xi^2-2(\gamma_1\alpha_1+\gamma_2\alpha_2)\zeta\xi=4(\gamma_1\alpha_2-\gamma_2\alpha_1)^2. \end{cases}$$

We can eliminate the α's and γ's from these equations by means of (81) and (85); we find, in fact,

$$(87)\quad \begin{cases} D\xi^2+E\eta^2+2F\xi\eta = J, \\ G\eta^2+D\zeta^2+2H\eta\zeta = K, \\ E\zeta^2+G\xi^2+2I\zeta\xi = L, \end{cases}$$

where

$$(88)\quad \begin{cases} D=P^2+Q^2, \quad E=R^2+S^2, \quad G=T^2, \\ F=QS-PR, \quad H=PT, \quad I=RT, \\ J=\frac{4}{D}\{PS+QR\}^2(\beta_1^2+\beta_2^2), \quad K=\frac{4}{D}Q^2T^2(\beta_1^2+\beta_2^2), \\ L=\frac{4}{D}S^2T^2(\beta_1^2+\beta_2^2). \end{cases}$$

7. Since at the centre of libration considered we have

$$(89)\qquad \Omega_{ab}=0, \quad \Omega_{ca}=0,$$

the quantities (88) become

$$(90)\quad \begin{cases} D=(\Omega_{aa}+\nu_2^2)^2\Omega_{bc}^2, \quad E=4\omega^2\nu_2^2\Omega_{bc}^2, \quad F=0, \\ G=\{(\Omega_{aa}+\nu_2^2)(\Omega_{bb}+\nu_2^2)-4\omega^2\nu_2^2\}^2, \quad H=\sqrt{(DG)}, \quad I=0, \\ J=4E(\beta_1^2+\beta_2^2), \quad K=0, \quad L=\frac{4EG}{D}(\beta_1^2+\beta_2^2)=\frac{JG}{D}; \end{cases}$$

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and we have the equations to the orbits in the form

$$(91)\qquad \begin{cases} D\xi^2 + E\eta^2 = J, \\ G\eta^2 + D\zeta^2 + 2\sqrt{(DG)}\,\eta\zeta = 0, \\ E\zeta^2 + G\xi^2 = \dfrac{JG}{D}; \end{cases}$$

accordingly the projections of the orbits are

$$(92)\qquad \begin{cases} \xi\eta\text{-plane} & \text{ellipse} & \dfrac{\xi^2}{\frac{J}{D}} + \dfrac{\eta^2}{\frac{J}{E}} = 1 \\ \eta\zeta\text{-plane} & \text{coincident lines} & (\sqrt{G}\,\eta + \sqrt{D}\,\zeta)^2 = 0, \\ \zeta\xi\text{-plane} & \text{ellipse} & \dfrac{\zeta^2}{\frac{JG}{DE}} + \dfrac{\xi^2}{\frac{J}{D}} = 1. \end{cases}$$

The minor axis is $2\sqrt{\left(\frac{J}{D}\right)}$ directed along the ξ-coordinate axis; the major axis lies in the $\zeta\eta$-plane, and is equal to $2\sqrt{\left\{\frac{J}{E}\left(1+\frac{G}{D}\right)\right\}}$; the centre of the orbit coincides with the centre of libration; the orbital plane passes through the ξ-axis, and makes an angle with the $\xi\eta$-plane whose tangent is equal to $-\sqrt{\left(\frac{G}{D}\right)}$.

The inclination of the orbits of the family is constant since it is independent of the arbitrary constants β_1 and β_2.

The numerical data (61), (62), (63) give

$$(93)\quad \log D = 2.452552,\ \log E = 2.373612,\ \log\sqrt{G} = 1.033029;$$

whence we find

$$(94)\qquad \begin{cases} \log\text{semi-minor axis} = 0.261560 + \log\sqrt{(\beta_1^2 + \beta_2^2)}, \\ \log\text{semi-major axis} = 0.375745 + \log\sqrt{(\beta_1^2 + \beta_2^2)}, \\ \log\tan\phi = 9.806754. \end{cases}$$

Designating by T and π the periods of the finite and

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infinitesimal bodies respectively, we have

$$T = \frac{2\pi}{\omega}, \quad \tau = \frac{2\pi}{\nu}; \tag{95}$$

hence

$$\frac{\tau}{T} = \frac{\sqrt{(m_1 + m_2 + m_3)}}{\nu} = \sqrt{\left(\frac{3}{4.42703}\right)} = 0.8232. \tag{96}$$

8. Before considering the motion when the infinitesimal body is in the plane of the finite bodies, it may be interesting to refer the problem to axes of more general position which admit of the presence of displacements in three independent directions, and accordingly are capable of revealing orbits in space about the centres of libration in the plane of the finite bodies. An example of such a system which preserves the symmetry and verifies the results of the previous analysis of this note is given below.

Take a system of rectangular axes in space and place the three finite bodies at the points whose coordinates are respectively

$$\begin{cases} & x, & y, & z, \\ m_1, & \frac{\sqrt{2}}{2}, & 0, & 0, \\ m_2, & 0, & \frac{\sqrt{2}}{2}, & 0, \\ m_3, & 0, & 0, & \frac{\sqrt{2}}{2}. \end{cases} \tag{97}$$

The three bodies will then form an equilateral triangle whose side is unity. Let (x, y, z) be the position of the particle referred to the above axes.

The equations of motion of a particle (X, Y, Z) referred to axes rotating about an axis with components of rotation $\omega_1, \omega_2, \omega_3$, under the influence of forces whose potential is U, are

$$\begin{cases} \frac{du}{dt} - \omega_3 v + \omega_2 w = \frac{\partial U}{\partial X}, \\ \frac{dv}{dt} - \omega_1 w + \omega_3 u = \frac{\partial U}{\partial Y}, \\ \frac{dw}{dt} - \omega_2 u + \omega_1 v = \frac{\partial U}{\partial Z}, \end{cases} \tag{98}$$

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in which

$$(99)\qquad \begin{cases} u = \dfrac{dX}{dt} - \omega_3 Y + \omega_2 Z, \\ v = \dfrac{dY}{dt} - \omega_1 Z + \omega_3 X, \\ w = \dfrac{dZ}{dt} - \omega_2 X + \omega_1 Y. \end{cases}$$

In the problem in hand the axis of rotation is perpendicular to the plane of the finite bodies, and passes through their centre of inertia. The angular velocity of the system in rotation about this axis is

$$(100)\qquad \omega = \sqrt{(m_1 + m_2 + m_3)};$$

the components of this angular velocity about axes parallel to the axes Ox, Oy, Oz above are

$$(101)\qquad \omega_1 = \omega_2 = \omega_3 = \omega \cos\cos^{-1}\sqrt{\tfrac{1}{3}} = \sqrt{\left(\frac{m_1+m_2+m_3}{3}\right)} = n \text{ (say)}.$$

If the origin be taken at the centre of inertia of the system and the axes parallel to Ox, Oy, Oz, the coordinates of the particle are

$$(102)\qquad X = x - \frac{\sqrt{2}}{2} m_1/\Sigma m, \quad Y = y - \frac{\sqrt{2}}{2} m_2/\Sigma m,$$

$$Z = z - \frac{\sqrt{2}}{2} m_3/\Sigma m,$$

and the equations of motion become

$$(103)\qquad \begin{cases} \dfrac{d^2x}{dt^2} - 2n\left(\dfrac{dy}{dt} - \dfrac{dz}{dt}\right) = n^2(2x - y - z) + \dfrac{\sqrt{2}}{6}(-2m_1 + m_2 + m_3) + \dfrac{\partial U}{\partial x}, \\ \dfrac{d^2y}{dt^2} - 2n\left(\dfrac{dz}{dt} - \dfrac{dx}{dt}\right) = n^2(-x + 2y - z) + \dfrac{\sqrt{2}}{6}(m_1 - 2m_2 + m_3) + \dfrac{\partial U}{\partial y}, \\ \dfrac{d^2z}{dt^2} - 2n\left(\dfrac{dx}{dt} - \dfrac{dy}{dt}\right) = n^2(-x - y + 2z) + \dfrac{\sqrt{2}}{6}(m_1 + m_2 - 2m_3) + \dfrac{\partial U}{\partial z}, \end{cases}$$

in which

$$(104)\qquad V = \frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} + \frac{m_3}{\rho_3},$$

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the ρ's designating the distances of the particle from the finite bodies.

Putting

$$(105)\qquad \Omega = n^2(x^2+y^2+z^2-xy-yz-zx)+\frac{\sqrt{2}}{6}\{(-2m_1+m_2+m_3)\,x + (m_1-2m_2+m_3)\,y+(m_1+m_2-2m_3)\,z\}+V,$$

the equations (103) become

$$(106)\qquad \begin{cases} \dfrac{d^2x}{dt^2}-2n\left(\dfrac{dy}{dt}-\dfrac{dz}{dt}\right)=\dfrac{\partial\Omega}{\partial x}, \\ \dfrac{d^2y}{dt^2}-2n\left(\dfrac{dz}{dt}-\dfrac{dx}{dt}\right)=\dfrac{\partial\Omega}{\partial z}. \\ \dfrac{d^2z}{dt^2}-2n\left(\dfrac{dx}{dt}-\dfrac{dy}{dt}\right)=\dfrac{\partial\Omega}{\partial z}. \end{cases}$$

The function Ω can be expressed in terms of m_1, m_2, m_3, ρ_1, ρ_2, ρ_3 alone. In fact, we have

$$(107)\qquad \begin{cases} \rho_1^2=\left(x-\dfrac{\sqrt{2}}{2}\right)^2+y^2+z^2, \\ \rho_2^2=x^2+\left(y-\dfrac{\sqrt{2}}{2}\right)^2+z^2, \\ \rho_3^2=x^2+y^2+\left(z-\dfrac{\sqrt{2}}{2}\right)^2, \end{cases}$$

which solved for x, y, z give

$$(108)\qquad \begin{cases} x=-\dfrac{\sqrt{2}}{6}(2\rho_1^2-\rho_2^2-\rho_3^2-1)\pm\tfrac{1}{3}\sqrt{(-R)}, \\ y=-\dfrac{\sqrt{2}}{6}(2\rho_2^2-\rho_3^2-\rho_1^2-1)\pm\tfrac{1}{3}\sqrt{(-R)}, \\ z=-\dfrac{\sqrt{2}}{6}(2\rho_3^2-\rho_1^2-\rho_2^2-1)\pm\tfrac{1}{3}\sqrt{(-R)}, \end{cases}$$

in which

$$(109)\quad R=\rho_1^2(\rho_1^2-\rho_2^2-1)+\rho_2^2(\rho_2^2-\rho_3^2-1)+\rho_3^2(\rho_3^2-\rho_1^2-1)+1.$$

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The irrational part of the values (108) causes no inconvenience, for, putting for short

$$x = \alpha \pm \mu, \quad y = \beta \pm \mu, \quad z = \gamma \pm \mu, \tag{110}$$

we have

$$\left\{\begin{aligned} \Omega &= n^2\{(\alpha\pm\mu)(\alpha-\beta)+(\beta\pm\mu)(\beta-\gamma)+(\gamma\pm\mu)(\gamma-\alpha)\} \\ &\quad + \frac{\sqrt{2}}{6}\{(-2m_1+m_2+m_3)(\alpha\pm\mu) \\ &\quad + (m_1-2m_2+m_3)(\beta\pm\mu)+(m_1+m_2-2m_3)(\gamma\pm\mu)\}+V, \\ &= n^2\{\alpha(\alpha-\beta)+\beta(\beta-\gamma)+\gamma(\gamma-\alpha)\} \\ &+ \frac{\sqrt{2}}{6}\{(-2m_1+m_2+m_3)\alpha+(m_1-2m_2+m_3)\beta+(m_1+m_2-2m_3)\gamma\}+V; \end{aligned}\right. \tag{111}$$

hence

$$\left\{\begin{aligned} \Omega &= (m_1+m_2+m_3)\{\rho_1^2(\rho_1^2-\rho_2^2)+\rho_2^2(\rho_2^2-\rho_3^2)+\rho_3^2(\rho_3^2-\rho_1^2)\} \\ &\quad + m_1\left(\frac{6}{\rho_1}+2\rho_1^2-\rho_2^2-\rho_3^2\right)+m_2\left(\frac{6}{\rho_2}+2\rho_2^2-\rho_3^2-\rho_1^2\right) \\ &\quad + m_3\left(\frac{6}{\rho_3}+2\rho_3^2-\rho_1^2-\rho_2^2\right); \end{aligned}\right. \tag{112}$$

this form and (11) are identical, as is apparent on rearranging (112).

Now let (a, b, c) be a centre of libration, and ξ, η, ζ the coordinates of the particle referred to it as origin; then

$$x = a + \xi, \quad y = b + \eta, \quad z = c + \zeta, \tag{113}$$

and the equations of motion yield the equations

$$\frac{\partial\Omega}{\partial a} = 0, \quad \frac{\partial\Omega}{\partial b} = 0, \quad \frac{\partial\Omega}{\partial c} = 0, \tag{114}$$

to determine the positions of the centres of libration. To evaluate them we have the following relations:

$$\rho_i^2 = (a-x_i)^2+(b-y_i)^2+(c-z_i)^2, \quad i = 1, 2, 3; \tag{115}$$

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$$(116)\qquad \begin{cases} x_1 = \dfrac{\sqrt{2}\,(m_2+m_3)}{2\Sigma m}, & x_2 = x_3 = -\dfrac{\sqrt{2}\,m_1}{2\Sigma m}, \\ y_1 = y_3 = -\dfrac{\sqrt{2}\,m_2}{2\Sigma m}, & y_2 = \dfrac{\sqrt{2}\,(m_3+m_1)}{2\Sigma m}, \\ z_1 = z_2 = -\dfrac{\sqrt{2}\,m_3}{2\Sigma m}, & y_3 = \dfrac{\sqrt{2}\,(m_1+m_2)}{2\Sigma m}. \end{cases}$$

We have further

$$(117)\qquad \frac{\partial\Omega}{\partial a} \equiv \sum_1^3{}_j \frac{a-x_j}{\rho_j}\frac{\partial\Omega}{\partial\rho_j}, \qquad \frac{\partial\Omega}{\partial b} \equiv \sum_1^3{}_j \frac{b-y_j}{\rho_j}\frac{\partial\Omega}{\partial\rho_j},$$

$$\frac{\partial\Omega}{\partial c} \equiv \sum_1^3{}_j \frac{c-z_j}{\rho_j}\frac{\partial\Omega}{\partial\rho_j};$$

hence the equations (114) demand either 1°,

$$(118)\qquad \begin{vmatrix} \dfrac{a-x_1}{\rho_1}, & \dfrac{a-x_2}{\rho_2}, & \dfrac{a-x_3}{\rho_3} \\ \dfrac{b-y_1}{\rho_1}, & \dfrac{b-y_2}{\rho_2}, & \dfrac{b-y_3}{\rho_3} \\ \dfrac{c-z_1}{\rho_1}, & \dfrac{c-z_2}{\rho_2}, & \dfrac{c-z_3}{\rho_3} \end{vmatrix} = 0;$$

or 2°,

$$(119)\qquad \frac{\partial\Omega}{\partial\rho_1} = \frac{\partial\Omega}{\partial\rho_2} = \frac{\partial\Omega}{\partial\rho_3} = 0.$$

The condition 1° can be written

$$(120)\qquad \frac{1}{\rho_1\rho_2\rho_3}\begin{vmatrix} a, & x_1, & x_2, & x_3 \\ b, & y_1, & y_2, & y_3 \\ c, & z_1, & z_2, & z_3 \\ 1, & 1, & 1, & 1, \end{vmatrix} = 0,$$

since no ρ can be infinite this demands that the determinant be zero, which expresses that the infinitesimal body lies in the plane of the finite bodies.

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The condition 2° above gives

$$(121)\quad \begin{cases} 3\dfrac{\partial\Omega}{\partial\rho_1} \equiv \left(2\rho_1^3 - \rho_1\rho_2^2 - \rho_1\rho_3^2 + 2\rho_1 - \dfrac{3}{\rho_1^2}\right) m_1 \\ \qquad\qquad + (2\rho_1^3 - \rho_1\rho_3^2 - \rho_1\rho_2^2 - \rho_1)\,(m_2 + m_3) = 0, \\ 3\dfrac{\partial\Omega}{\partial\rho_2} \equiv (2\rho_2^3 - \rho_1^2\rho_2 - \rho_2\rho_3^2 - \rho_2)\,(m_1 + m_2) \\ \qquad\qquad + \left(2\rho_2^3 - \rho_1^2\rho_2 - \rho_3^2\rho_2 + 2\rho_2 - \dfrac{3}{\rho_2^2}\right) m_2 = 0, \\ 3\dfrac{\partial\Omega}{\partial\rho_3} \equiv (2\rho_3^3 - \rho_1^2\rho_3 - \rho_2^2\rho_3 - \rho_3)\,(m_1 + m_2) \\ \qquad\qquad + \left(2\rho_3^3 - \rho_1^2\rho_3 - \rho_2^2\rho_3 + 2\rho_3 - \dfrac{3}{\rho_3^2}\right) m_3 = 0. \end{cases}$$

Let the masses be taken equal, the last equations then become

$$(122)\quad X^3(2X^3 - Y^3 - Z^3) = Y^3(2Y^3 - Z^3 - X^3) = Z^3(2Z^3 - Y^3 - X^3) = 1,$$

replacing ρ_1, ρ_2, ρ_3 respectively by X, Y, Z; these equations (122) are equivalent to

$$(123)\quad \begin{cases} X^3Y^3 + Y^3Z^3 + Z^3X^3 = 0, \\ 3X^3Y^6 - 3X^6Y^3 - X^3 + Y^3 = 0, \\ 3X^3Z^6 - 3X^6Z^3 - X^3 + Z^3 = 0, \end{cases}$$

which are identical with equations (38), (39) and (40).

The equations of motion of the particle in the immediate vicinity of a centre of libration

$$(124)\quad \begin{cases} \dfrac{d^2\xi}{dt^2} - 2n\left(\dfrac{d\eta}{dt} - \dfrac{d\zeta}{dt}\right) = \xi\Omega_{aa} + \eta\Omega_{ab} + \zeta\Omega_{ca}, \\ \dfrac{d^2\eta}{dt^2} - 2n\left(\dfrac{d\zeta}{dt} - \dfrac{d\xi}{dt}\right) = \xi\Omega_{ab} + \eta\Omega_{bb} + \zeta\Omega_{bc}, \\ \dfrac{d^2\zeta}{dt^2} - 2n\left(\dfrac{d\xi}{dt} - \dfrac{d\eta}{dt}\right) = \xi\Omega_{ca} + \eta\Omega_{bc} + \zeta\Omega_{cc}, \end{cases}$$

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constitute a system of linear differential equations with constant coefficients. It possesses a solution of the form

$$\xi = Ae^{\lambda t}, \quad \eta = Be^{\lambda t}, \quad \zeta = Ce^{\lambda t}, \tag{125}$$

in which the mutual ratios of A, B, C are found from the equations

$$\begin{cases} (\lambda^2 - \Omega_{aa})A - (2n\lambda + \Omega_{ab})B + (2n\lambda - \Omega_{ca})C = 0, \\ (2n\lambda - \Omega_{ab})A + (\lambda^2 - \Omega_{bb})B - (2n\lambda + \Omega_{bc})C = 0, \\ -(2n\lambda + \Omega_{ca})A + (2n\lambda - \Omega_{bc})B + (\lambda^2 - \Omega_{cc})C = 0; \end{cases} \tag{126}$$

and λ is a root of the equation

$$\begin{vmatrix} \lambda^2 - \Omega_{aa}, & -2n\lambda - \Omega_{ab}, & 2n\lambda - \Omega_{ca} \\ 2n\lambda - \Omega_{ab}, & \lambda^2 - \Omega_{bb}, & -2n\lambda - \Omega_{bc} \\ -2n\lambda - \Omega_{ca}, & 2n\lambda - \Omega_{bc}, & \lambda^2 - \Omega_{cc} \end{vmatrix} = 0, \tag{127}$$

or in expanded form

$$\lambda^6 + k_1\lambda^4 + k_2\lambda^2 + k_3 = 0, \tag{128}$$

in which

$$\begin{cases} k_1 = 12n^2 - \Omega_{aa} - \Omega_{bb} - \Omega_{cc}, \\ k_2 = \Omega_{aa}\Omega_{bb} + \Omega_{bb}\Omega_{cc} + \Omega_{cc}\Omega_{aa} - 8n^2(\Omega_{ab} + \Omega_{bc} + \Omega_{ca}) \\ \qquad - 4n^2(\Omega_{aa} + \Omega_{bb} + \Omega_{cc}) - \Omega_{ab}^2 - \Omega_{bc}^2 - \Omega_{ca}^2 \\ k_3 = \Omega_{aa}\Omega_{bc}^2 + \Omega_{bb}\Omega_{ca}^2 + \Omega_{cc}\Omega_{ab}^2 - \Omega_{aa}\Omega_{bb}\Omega_{cc} - 2\Omega_{ab}\Omega_{bc}\Omega_{ca}. \end{cases} \tag{129}$$

Solving the equations (115) with special reference to the points for which

$$\rho_2^2 = \rho_1^2, \tag{130}$$

we have

$$a = b = \alpha \pm \beta, \quad c = -2\alpha \pm \beta, \tag{131}$$

where

$$\alpha = \frac{\rho_2^2 - \rho_1^2}{3\sqrt{2}}, \quad \beta = \tfrac{1}{3}\sqrt{\{3\rho_2^2 - (\rho_2^2 - \rho_3^2 - 1)^2\}}; \tag{132}$$

further, for unit masses, the second derivatives of Ω may be

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written

$$(133)\quad \begin{cases} \Omega_{aa}=\sum_1^3 i\ (a-x_i)^2\phi_i-4\psi(a,\ x),\ \ \Omega_{bb}=\sum_1^3 i\ (b-y_i)^2\phi_i-4\psi(b,\ y), \\ \Omega_{cc}=\sum_1^3 i\ (c-z_i)^2\phi_i-4\psi\ (c,\ z), \\ \Omega_{ab}=\alpha_{123}+\alpha_{231}+\alpha_{312}, \qquad \Omega_{bc}=\beta_{123}+\beta_{231}+\beta_{312}, \\ \Omega_{ca}=\gamma_{123}+\gamma_{231}+\gamma_{312}, \end{cases}$$

where

$$(134)\quad \begin{cases} \phi_i=4+\dfrac{3}{\rho_i^5},\quad \psi\ (k,\ l)=(k-l_1)\ (k-l_2) \\ \qquad\qquad +(k-l_2)\ (k-l_3)+(k-l_3)\ (k-l_1), \\ \alpha_{ijk}=(a-x_i)\ \{\phi_i\ (b-y_i)-2\ (b-y_j)-2\ (b-y_k)\}, \\ \beta_{ijk}=(b-y_i)\ \{\phi_i\ (c-z_i)-2\ (c-z_j)-2\ (c-z_k)\}, \\ \gamma_{ijk}=(c-z_i)\ \{\phi_i\ (a-x_i)-2\ (a-x_j)-2\ (a-x_k)\}. \end{cases}$$

Taking

$$(135)\qquad \rho_1=\rho_2=1.21997,\quad \rho_3=-0.96829,$$

the values (131) are

$$(136)\quad a=b=-0.12982\pm 0.53862,\ \ c=+0.25964\pm 0\ 53862;$$

putting the first of these two sets of coordinates in (133), we find

$$(137)\quad \begin{cases} \Omega_{aa}=\Omega_{bb}=2.99010, \qquad \Omega_{cc}=0\ 01962, \\ \Omega_{ab}=-0.56492,\quad \Omega_{bc}=\Omega_{ca}=-2.06890; \end{cases}$$

these, together with unity as the value of n, give for the characteristic equation in λ

$$(138)\qquad \lambda^6+6.0002\lambda^4+13.8007\lambda^2+30.2644=0,$$

which agree with the previous results (64) and (65).

In precisely the same manner as in the preceding discussion, we find equations to orbits in the form

$$(139)\quad \begin{cases} D\xi^2+E\eta^2+2F\xi\eta=J, \\ G\eta^2+D\zeta^2+2H\eta\zeta=K, \\ E\zeta^2+G\xi^2+2I\zeta\xi=L, \end{cases}$$

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with

$$(140)\quad \begin{cases} D=P^2+Q^2,\ E=R^2+S^2,\ G=T^2,\ F=QS-PR, \\ H=PT,\ I=RT, \\ J=\frac{4}{D}\{PS+QR\}^2(\beta_1^2+\beta_2^2),\quad K=\frac{4}{D}Q^2T^2(\beta_1^2+\beta_2^2), \\ L=\frac{4}{D}S^2T^2(\beta_1^2+\beta_2^2), \end{cases}$$

in which

$$(141)\quad \begin{cases} P=(4n^2+\Omega_{bc})\,\nu_2^2+\Omega_{aa}\Omega_{bc}-\Omega_{ab}\Omega_{ca}, \\ Q=2n\nu_2^3+2n\nu_2(\Omega_{aa}+\Omega_{ab}+\Omega_{ca}), \\ R=(4n^2+\Omega_{ca})\,\nu_2^2+\Omega_{bb}\Omega_{ca}-\Omega_{ab}\Omega_{bc}, \\ S=2n\nu_2^3+2n\nu_2(\Omega_{ab}+\Omega_{bb}+\Omega_{bc}), \\ T=(\nu_2^2+\Omega_{aa})(\nu_2^2+\Omega_{bb})-\Omega_{ab}^2-4n^2\nu_2^2. \end{cases}$$

At the centre of libration considered, we have

$$(142)\qquad \Omega_{aa}=\Omega_{bb},\quad \Omega_{bc}=\Omega_{ca},$$

hence

$$(143)\qquad R=P,\quad S=Q,$$

and (140) become

$$(144)\quad \begin{cases} D=E=P^2+Q^2,\quad F=Q^2-P^2,\quad G=T^2, \\ H=I=PT,\qquad J=\frac{16}{D}P^2Q^2(\beta_1^2+\beta_2^2), \\ K=L=\frac{4}{D}Q^2T^2(\beta_1^2+\beta_2^2); \end{cases}$$

the equations to the orbits then become

$$(145)\quad \begin{cases} (P^2+Q^2)(\xi^2+\eta^2)+2(Q^2-P^2)\,\xi\eta=\frac{16P^2Q^2}{P^2+Q^2}(\beta_1^2+\beta_2^2), \\ T^2\eta^2+(P^2+Q^2)\,\zeta^2+2PT\eta\zeta=\frac{4Q^2T^2}{P^2+Q^2}(\beta_1^2+\beta_2^2), \\ (P^2+Q^2)\,\zeta^2+T^2\xi^2+2PT\zeta\xi=\frac{4Q^2T^2}{P^2+Q^2}(\beta_1^2+\beta_2^2); \end{cases}$$

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remarking finally that we pass from the axes of the first problem of this note to those used here by means of the scheme of direction-cosines,

(146)

	x'	y'	z'
x	$\sqrt{\frac{2}{3}}$	0	$\sqrt{\frac{1}{3}}$
y	$-\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$
z	$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$

the identity of the orbits (145) with those given by equations (92) is easily made manifest.

9. Returning now to the case where c is equal to zero and the four points coplanar, we have the following equations to determine the positions of the centres of libration in the plane of the three finite bodies:

(147)
$$\begin{cases} \sum_1^3 \frac{\partial \Omega}{\partial \rho_j} \frac{a - x_j}{\rho_j} = 0, \\ \sum_1^3 \frac{\partial \Omega}{\partial \rho_j} \frac{b - y_j}{\rho_j} = 0, \\ \rho_1^2(\rho_1^2 - \rho_2^2 - 1) + \rho_2^2(\rho_2^2 - \rho_3^2 - 1) + \rho_3^2(\rho_3^2 - \rho_1^2 - 1) + 1 = 0, \end{cases}$$

in which

(148)
$$\frac{\partial \Omega}{\partial \rho_j} = m_j \left(\rho_j - \frac{1}{\rho_j^2} \right).$$

Recalling that

(149)
$$\begin{cases} x_1 = -x_2 = -\frac{1}{2}, \quad x_3 = 0, \\ y_1 = y_2 = \frac{-\sqrt{3}}{6}, \quad y_3 = \frac{\sqrt{3}}{3}, \end{cases}$$

we find in the case of three equal masses of unit value that the equations (147) possess the following real solutions, using six-place logarithm tables in effecting the calculations:

(150)
$$\begin{cases} (0, 0), \ (0, -0.2390), \ (0, -0.9352), \ (0, +1.1800), \\ (\pm 0.2070, \ +0.1195), \ (\pm 0.8099, \ +0.4676), \\ (\pm 1.0219, \ -0.5900). \end{cases}$$

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These points lie on the axes of symmetry of the equilateral triangle, and on three concentric circles about the origin as centre.

The familiar mechanism for determining periodic solutions of linear differential equations with constant coefficients as employed above, and as used generally, for example, more recently by Charlier (*Meddelanden fran Lunds Astronomiska Observatorium*, No. 18, 14 November, 1900), to design periodic orbits in the immediate vicinity of the centres of libration of the restricted problem of three bodies applies itself directly to the construction of the orbits sought here; in fact the formulæ for the three-body problem extend themselves immediately to the present problem, if the necessary terms be added to Darwin's form of the force-function.

The characteristic equation becomes now the quadratic in λ^2,

$$\lambda^4 + (4\omega^2 - \Omega_{aa} - \Omega_{bb})\lambda^2 + \Omega_{aa}\Omega_{bb} - \Omega_{ab}^2 = 0, \tag{151}$$

where

$$\Omega = \sum_1^3 m_j \left(\rho_j^2 + \frac{2}{\rho_j}\right), \tag{152}$$

and accordingly

$$\begin{cases} \Omega_{aa} = \sum_1^3 m_j \left\{1 - \frac{1}{\rho_j^3} + \frac{3}{\rho_j^5}(a - x_j)^2\right\}, \\ \Omega_{ab} = 3\sum_1^3 \frac{m_j}{\rho_j^5}(a - x_j)(b - y_j), \\ \Omega_{bb} = \sum_1^3 m_j \left\{1 - \frac{1}{\rho_j^3} + \frac{3}{\rho_j^5}(b - y_j)^2\right\}. \end{cases} \tag{153}$$

For convenience let the quadratic (151) be written

$$\lambda^4 + l_1\lambda^2 + l_2 = 0, \tag{154}$$

by putting

$$l_1 = 4\omega^2 - \Omega_{aa} - \Omega_{bb}, \quad l_2 = \Omega_{aa}\Omega_{bb} - \Omega_{ab}^2. \tag{155}$$

Using six-place logarithms and the elements (149), (150), (153), and (155) the following table is readily constructed, the accordance of the results being a satisfactory test of the accuracy of the computations:

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a	b	ρ_1	ρ_2	ρ_3	Ω_{aa}	Ω_{bb}	Ω_{ab}	l_1	l_2	λ_1^2	λ_2^2
0	0	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$	$\frac{3}{2}(2+3\sqrt{3})$	$\frac{3}{2}(2+5\sqrt{3})$	0	$3(2-3\sqrt{3})$	$\frac{9}{4}(2+3\sqrt{3})^2$	Imagi	nary
0	−0.2390	0.5024	0.5024	0.8163	32.25	−8.63	0	−11.62	278.44	Imagi	nary
0.2070	0.1195	0.8163	0.5024	0.5024	1.59	22.03	−17.70	−11.62	278.44	Imagi	nary
−0.2070	0.1195	0.5024	0.8163	0.5024	1.59	22.03	+17.70	−11.62	278.44	Imagi	nary
0	−0.9352	0.8163	0.8163	1.5125	2.16	5.79	0	4.05	12 51	Imagi	nary
0.8099	0.4676	1.5125	0.8163	0.8163	4.88	3.07	1.57	4.05	12.51	Imagi	nary
−0.8099	0.4676	0.8163	1.5125	0.8163	4.88	3.07	−1.57	4.05	12.51	Imagi	nary
0	1.1800	1.5515	1.5515	0.6027	−1.94	13.04	0	0.90	−25.24	4.59	−5.49
1.0219	−0.5900	1.5515	0.6027	1.5515	9.29	1.81	−6.48	0.90	−25.24	4.59	−5.49
−1.0219	−0.5900	0.6027	1.5515	1.5515	9.29	1.81	6.48	0.90	−25.24	4.59	−5.49

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We obtain without difficulty periodic solutions of the form

$$(157) \qquad \xi = 2\alpha_1 \cos \nu_2 t + 2\alpha_2 \sin \nu_2 t, \quad \eta = 2\beta_1 \cos \nu_2 t + 2\beta_2 \sin \nu_2 t,$$

where

$$(158) \qquad \begin{cases} (\Omega_{aa} + \nu_2^2)\,\alpha_1 = -\Omega_{ab}\beta_1 - 2\omega\nu_2\beta_2, \\ (\Omega_{aa} + \nu_2^2)\,\alpha_2 = 2\omega\nu_2\beta_1 - {}_{ab}\Omega\beta_2, \\ (\Omega_{aa} + \nu_2^2)\,(\Omega_{bb} + \nu_2^2) = 4\omega^2\nu_2^2 + \Omega_{ab}^2, \\ \qquad\qquad \nu^2 = -\lambda^2. \end{cases}$$

Eliminating t and the constants, the equation to the orbit appears in the form

$$(159) \quad (\nu_2^2 + \Omega_{aa})\,\xi^2 + (\nu_2^2 + \Omega_{bb})\,\eta^2 + 2\Omega_{ab}\xi\eta = \frac{16\omega^2\nu_2^2}{\nu_2^2 + \Omega_{aa}}(\beta_1^2 + \beta_2^2),$$

which is identical in form to that given by Charlier in the restricted problem of three bodies.

This curve is an ellipse about those points for which λ^2 is real and negative, since

$$(160) \qquad \Omega_{ab}^2 - (\nu_2^2 + \Omega_{aa})\,(\nu_2^2 + \Omega_{bb}) \equiv \Omega_{ab}^2 - 4\omega^2\nu_2^2 - \Omega_{ab}^2 < 0.$$

Designating the three points of (156) about which periodic orbits are possible by

$$(161) \qquad L_1\,(0,\ 1.1800), \quad L_2\,(1.0219,\ -0.5900),$$

$$L_3\,(-1.0219,\ -0.5900);$$

the ellipses about L_1 have the equations

$$(162) \qquad \frac{\xi^2}{12\sqrt{3}} + \frac{\eta^2}{4} = \beta_1^2 + \beta_2^2;$$

those about L_2 and L_3 constitute two families congruent to (162) as appears by referring them to their own axes by rotation through the angle θ defined by the equation

$$(163) \qquad \tan 2\theta = \frac{2\Omega_{ab}}{\Omega_{aa} - \Omega_{ab}} = \pm\sqrt{3}.$$

Letting the initial position of (ξ, η) at $t = 0$ be (ξ_0, η_0), we

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have from equations (157) and (158)

$$2\omega\eta_0' = -(\nu_2^2 + \Omega_{aa})\,\xi_0 - \Omega_{a\beta}\eta_0, \tag{164}$$

where the prime indicates differentiation with regard to t.

Taking the origin of time at the instant when the particle is at $(\xi_0, 0)$, and remarking that the quantity $(\nu_2^2 + \Omega_{aa})$ is positive for all three orbits, we see that the orbits are described in a direction, observed from the corresponding centre, opposite to that of the motion of the finite bodies.

The periods of the orbits about the points (161) are identical and the common period is

$$\sqrt{\left(\frac{3}{5.49}\right)}\,T = 0.74\,T, \tag{165}$$

where T is the period of the finite bodies about their centre of gravity.

10. Among the questions presenting themselves in this connection, there are several which it is hoped to study in detail in a subsequent note; for example, 1° the Jacobian integral, the characteristic equation and its roots for a range of values of the masses of the finite bodies within the limits of stability; 2° a more rigorous integration of the equations of motion of the particle; 3° the same problems when the finite bodies move in conics; 4° the orbits in general under the most arbitrary choice of constants including those in space; 5° singular solutions and those trajectories on which collisions are possible.

As remarked in the introductory paragraph of this note the writer has adapted a method of Levi-Civita in the restricted problem of three bodies to the cases of the problem of four bodies considered above; these results are reserved for another paper.

In the preceding pages we have treated the case in which the finite bodies constitute an equiangular triangle solution of Lagrange. In conclusion we shall construct periodic solutions corresponding to the rectilinear solution of Lagrange as generalised by Lehmann-Filhés for n bodies, and in fact for any values of the masses and initial orbits of circles or conic sections. This section of the note consists of extensions to n bodies of results given for the case of three bodies by Oppenheim in the memoir to which reference has already been made. These extensions are made with the greatest ease as the

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reader will observe, and the results are appended here only because of their interest in this connection.

11. Consider the motion of n bodies under the law of Newton; the equations of motion may be written

$$(166)\quad \begin{cases} m_i\dfrac{d^2x_i}{dt^2}+m_ix_i\sum\limits_1^n{}_j\dfrac{m_j}{\rho_{ij}^3}-m_i\sum\limits_1^n{}_j\dfrac{m_jx_j}{\rho_{ij}^3}=0, \\ m_i\dfrac{d^2y_i}{dt^2}+m_iy_i\sum\limits_1^n{}_j\dfrac{m_j}{\rho_{ij}^3}-m_i\sum\limits_1^n{}_j\dfrac{m_jy_j}{\rho_{ij}^3}=0, \\ m_i\dfrac{d^2z_i}{dt^2}+m_iz_i\sum\limits_1^n{}_j\dfrac{m_j}{\rho_{ij}^3}-m_i\sum\limits_1^n{}_j\dfrac{m_jz_j}{\rho_{ij}^3}=0, \end{cases} \quad i=1,2,\ldots,n;\ j\neq i.$$

and if the axes be taken through the centre of gravity the integrals of the centre of gravity become

$$(167)\qquad \sum_1^n{}_i\, m_ix_i=\sum_1^n{}_i\, m_iy_i=\sum_1^n{}_i\, m_iz_i=0.$$

The equations for the integrals of areas are

$$(168)\quad \sum_1^n{}_i\, m_i\left(x_i\frac{d^2y_i}{dt^2}-y_i\frac{d^2x_i}{dt^2}\right)=\sum_1^n{}_i\, m_i\left(y_i\frac{d^2z_i}{dt^2}-z_i\frac{d^2y_i}{dt^2}\right)$$
$$=\sum_1^n{}_i\, m_i\left(z_i\frac{d^2x_i}{dt^2}-x_i\frac{d^2z_i}{dt^2}\right)=0.$$

In virtue of (166) and (167) the elements of the summations in (168) may be written

$$(169)\quad \begin{cases} m_i\left(x_i\dfrac{d^2y_i}{dt^2}-y_i\dfrac{d^2x_i}{dt^2}\right)=m_i\sum\limits_1^{n-1}{}_j\, m_j(x_iy_j-x_jy_i)\left(\dfrac{1}{\rho_{ij}^3}-\dfrac{1}{\rho_{in}^3}\right), \\ m_i\left(x_i\dfrac{d^2z_i}{dt^2}-z_i\dfrac{d^2x_i}{dt^2}\right)=m_i\sum\limits_1^{n-1}{}_j\, m_j(y_iz_j-z_iy_j)\left(\dfrac{1}{\rho_{ij}^3}-\dfrac{1}{\rho_{in}^3}\right), \\ m_i\left(z_i\dfrac{d^2x_i}{dt^2}-x_i\dfrac{d^2z_i}{dt^2}\right)=m_i\sum\limits_1^{n-1}{}_j\, m_j(z_ix_j-z_jx_i)\left(\dfrac{1}{\rho_{ij}^3}-\dfrac{1}{\rho_{in}^3}\right); \end{cases} \quad i=1,2,\ldots,n;\ j\neq i;$$

these elements are severally equal to zero in either of the following cases:—

1°. $x_iy_j-x_jy_i=y_iz_j-y_jz_i=z_ix_j-z_jx_i=0$, $i,j=1,2,\ldots,n$;

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or

$$2^{\circ}. \qquad \rho_{ij} = \rho_{kl}, \quad i, j, k, l = 1, 2, \ldots, n.$$

The first of these require that the n bodies lie on a straight line; the second demand that they be located at the vertices of a regular polyhedron, the latter in ordinary space is naturally limited to $n \leq 4$. These results lead to the integrable cases of the problem of n bodies found by Lehmann-Filhés ("Ueber zwei Fälle des Vielkörperproblems, *Astronomische Nachrichten*, Vol. 127, 1891). The possibility of this extension was remarked but not effected by Oppenheim.

12. Let us integrate the equations of motion under the first of these conditions; the second condition cannot lead to periodic solutions for $n > 3$.

The first condition

$$\rho_{12} \pm \rho_{23} \pm \ldots \pm \rho_{n-1,n} \pm \rho_{n1} = 0 \tag{170}$$

can be handled for all values of n.

Assume some order of magnitude merely for purposes of reference, say

$$m_1 \geq m_2 \geq m_3 \geq \ldots \geq m_{n-1} \geq m_n \tag{171}$$

and let the centre of gravity lie between m_{k-1} and m_k.

Calling r_i the distance of m_i from the centre of gravity, we have

$$\begin{cases} r_i - r_j = \rho_{ij}, \ i, j = 1, 2, \ldots, k-1; \\ r_{k-1} + r_k = \rho_{k-1,k}; \\ r_j - r_i = \rho_{ij+1}, \ i, j = k+1, k+2, \ldots, n. \end{cases} \tag{172}$$

Introducing the polar coordinates

$$x_i = r_i \cos\phi, \quad y_i = r_i \sin\phi, \quad i = 1, 2, \ldots, n, \tag{173}$$

in consequence of the equations

$$r_i^2 \frac{d\phi}{dt} = c_i, \ i = 1, 2, \ldots, n;$$

we must have

$$r_i = \alpha_i \rho, \quad i = 1, 2, \ldots, n, \tag{174}$$

where the α_i are constants.

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These values of r_i give

$$(175) \qquad x_i = \pm \alpha_i \rho \cos\phi, \; y_i = \pm \alpha_i \rho \sin\phi, \; \rho_{ij} = (\alpha_i \pm \alpha_j)\,\rho;$$

where the ambiguity in sign is readily removed, when necessary, by the aid of the above conventions.

Substituting the values (175) in the equations of motion, we have

$$(176) \quad \begin{cases} \alpha_i \dfrac{d^2}{dt^2}(\rho\cos\phi) + \dfrac{\cos\phi}{\rho^2} \sum_1^n{}^j \dfrac{m_j}{(\alpha_i \pm \alpha_j)^2} = 0, \\ \alpha_i \dfrac{d^2}{dt^2}(\rho\sin\phi) + \dfrac{\sin\phi}{\rho^2} \sum_1^n{}^j \dfrac{m_j}{(\alpha_i \pm \alpha_j)^2} = 0, \end{cases} \quad i, j = 1, 2, \ldots, n, j \neq i.$$

These equations must exist identically and hence give the following equations

$$(177) \qquad \sum_1^n{}^j \frac{m_j}{(\alpha_i \pm \alpha_j)^2} = \alpha_i k^2, \quad i = 1, 2, \ldots, n; \; j \neq 1.$$

The equations for ρ and ϕ then become

$$(178) \qquad \frac{d^2}{dt^2}(\rho\cos\phi) + \frac{k^2\cos\phi}{\rho^2} = 0, \quad \frac{d^2}{dt^2}(\rho\sin\phi) + \frac{k^2\sin\phi}{\rho^2} = 0.$$

These equations have the same form as those for ordinary elliptic motion, and hence their integrals are

$$(179) \qquad \rho^2 \frac{d\phi}{dt} = c, \; \rho = \frac{p}{1 + e\cos\phi}, \; c = k\sqrt{p}, \; n = ka^{-\frac{3}{2}},$$

which express that the orbits are conic sections with a common focus at the centre of gravity of the system.

The quantities k^2 and α_i can be determined from the equations (177), since one of the α_i is arbitrary.

13. To obtain other periodic solutions put

$$(180) \qquad x_i = x_{i0} + \mu\xi_i, \quad y_i = y_{i0} + \mu\eta_i, \quad i = 1, 2, \ldots, n,$$

where μ is a small quantity.

Let the force-function be $-F$, where

$$(181) \qquad F = -\sum_1^n{}^i \, \Sigma^j \frac{m_i m_j}{\rho_{ij}}, \quad j \neq i;$$

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and the equations of motion become

$$(182)\qquad m_i\frac{d^2x_i}{dt^2}+\frac{\partial F}{\partial x_i}=m_i\frac{d^2y_i}{dt^2}+\frac{\partial F}{\partial y_i}=0,\quad i=1,\,2,\,\ldots,\,n.$$

Then the quantities ξ_i and η_i satisfy the equations

$$(183)\qquad \begin{cases} m_i\dfrac{d^2\xi_i}{dt^2}+\sum_1^n{}_j\left(\xi_j\dfrac{\partial^2F}{\partial x_i\partial x_j}+\eta_j\dfrac{\partial^2F}{\partial x_i\partial y_j}\right)=0, \\ m_i\dfrac{d^2\eta_i}{dt^2}+\sum_1^n{}_j\left(\xi_j\dfrac{\partial^2F}{\partial y_i\partial x_j}+\eta_j\dfrac{\partial^2F}{\partial y_i\partial y_j}\right)=0, \end{cases} \quad i=1,\,2,\,\ldots,\,n,$$

From equations (175) the differential coefficients in these equations have the forms

$$(184)\qquad \begin{cases} F_{xx_i}=\dfrac{1}{\rho^3}(1-3\cos^2\phi)\sum_1^n{}_j\dfrac{m_im_j}{(\alpha_i\pm\alpha_j)^3}, \\ F_{xx_j}=-\dfrac{1}{\rho^3}(1-3\cos^2\phi)\dfrac{m_im_j}{(\alpha_i\pm\alpha_j)^3}, \\ F_{yy_i}=\dfrac{1}{\rho^3}(1-3\sin^2\phi)\sum_1^n{}_j\dfrac{m_im_j}{(\alpha_i\pm\alpha_j)^3}, \quad i=1,\,2,\,\ldots,\,n;\ j\neq i; \\ F_{yy_j}=-\dfrac{1}{\rho^3}(1-3\sin^2\phi)\dfrac{m_im_j}{(\alpha_i\pm\alpha_j)^3}, \\ F_{xy_j}=\dfrac{1}{\rho^3}\sin\phi\cos\phi\dfrac{m_im_j}{(\alpha_i\pm\alpha_j)^3}. \end{cases}$$

On substituting these values in equations (183), we have a system of equations for the variations. The latter system can be written in convenient form by introducing the following abbreviations:

$$(185)\qquad \begin{cases} a_{11}=\sum_2^n{}_j\dfrac{m_j}{(\alpha_j\pm\alpha_1)^3},\ a_{12}=\dfrac{m_2}{(\alpha_2\pm\alpha_1)^3},\ \ldots,\ a_{1n}=\dfrac{m_n}{(\alpha_n\pm\alpha_1)^3},\ j\neq1, \\ a_{21}=\dfrac{m_1}{(\alpha_1\pm\alpha_2)^3},\ a_{22}=\sum_1^n{}_j\dfrac{m_j}{(\alpha_j\pm\alpha_2)^3},\ \ldots,\ a_{2n}=\dfrac{m_n}{(\alpha_n\pm\alpha_2)^3},\ j\neq2, \\ a_{31}=\dfrac{m_1}{(\alpha_1\pm\alpha_3)^3},\ a_{32}=\dfrac{m_2}{(\alpha_2\pm\alpha_3)^3},\ \ldots,\ a_{3n}=\dfrac{m_n}{(\alpha_n\pm\alpha_3)^3},\ j\neq3, \\ \cdots\cdots\cdots\

The equations (183) then become

$$(186)\quad \begin{cases} \rho^3 \dfrac{d^2\xi_i}{dt^2} + (1-3\cos^2\phi)(a_{ii}\xi_i - \sum_1^n{}' a_{ij}\xi_j \\ \qquad -3\sin\phi\cos\phi\,(a_{ii}\eta_i - \sum_1^n{}' a_{ij}\eta_j) = 0, \\ \qquad\qquad\qquad\qquad i=1, 2, \ldots, n;\ j\neq i; \\ \rho^3 \dfrac{d^2\eta_i}{dt^2} - 3\sin\phi\cos\phi\,(a_{ii}\xi_i - \sum_1^n{}' a_{ij}\xi_j) \\ \qquad +(1-3\sin^2\phi)\,(a_{ii}\eta_i - \sum_1^n{}' a_{ij}\eta_j) = 0. \end{cases}$$

This system of equations may be simplified by multiplying the first n equations by $A_1, A_2, \ldots, A_n$ respectively, determining these multipliers so that

$$(187)\qquad a_{ii}A_i - \sum_1^n{}' a_{ji}A_j = \lambda A_i,\quad i=1, 2, \ldots, n;\ j\neq i,$$

and putting

$$(188)\qquad \sum_1^n{}_i\, A_i\xi_i = X,\quad \sum_1^n{}_i\, A_i\eta_i = Y;$$

the system (186) becomes

$$(189)\quad \begin{cases} \rho^3 \dfrac{d^2X}{dt^2} + (1-3\cos^2\phi)\,\lambda X - (3\sin\phi\cos\phi)\,\lambda Y = 0, \\ \rho^3 \dfrac{d^2Y}{dt^2} + (1-3\sin^2\phi)\,\lambda Y - (3\sin\phi\cos\phi)\,\lambda X = 0; \end{cases}$$

to which must, of course, be appended

$$(190)\qquad \rho^2\frac{d\phi}{dt} = c,\quad \rho = \frac{p}{1+e\cos\phi},\quad c = k\sqrt{p},\quad \nu = ka^{-\frac{1}{2}}.$$

This reduction is possible if the equation for the determination of λ, namely the determinantal equation

$$(191)\qquad \Delta \equiv \begin{vmatrix} \lambda - a_{11}, & a_{21}, & a_{31}, \ldots, & a_{n1} \\ a_{12}, & \lambda - a_{22}, & a_{32}, \ldots, & a_{n2} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{1n}, & a_{2n}, & a_{3n}, \ldots, & \lambda - a_{nn} \end{vmatrix} = 0,$$

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has n unequal roots, for we shall have then n different sets of $A_1, A_2, \ldots, A_n$, and hence n different sets of X and as many of Y from which to determine the quantities $\xi_1, \xi_2, \ldots, \xi_n$; $\eta_1, \eta_2, \ldots, \eta_n$.

From (185) we have

$$a_{ii} - \sum_1^n{}_j a_{ij} = 0, \quad i = 1, 2, \ldots, n; \; j \neq i; \tag{192}$$

hence the determinant

$$\begin{vmatrix} -a_{11}, & a_{21}, & a_{31}, & \ldots, & a_{n1} \\ a_{12}, & -a_{22}, & a_{32}, & \ldots, & a_{n2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{1n}, & a_{2n}, & a_{3n}, & \ldots, & -a_{nn} \end{vmatrix} \tag{193}$$

vanishes identically; accordingly the equation (191) has a zero root.

In virtue of the relations

$$\sum_1^n{}_j m_j \alpha_{j1} = \sum_1^n{}_j m_j \alpha_{j2} = \ldots = \sum_1^n{}_j m_j a_{jn} = 0, \tag{194}$$

this zero value of λ corresponds to

$$A_1 = m_1, \quad A_2 = m_2, \quad \ldots, \quad A_n = m_n, \tag{195}$$

and gives the centre of gravity property

$$\sum_1^n{}_i m_i \xi_i = \sum_1^n{}_i m_i \eta_i = 0. \tag{196}$$

The equations (189) are identical in form with those of Oppenheim in the problem of three bodies and can be integrated in precisely the same manner.

Considering first the case of circular orbits, we have

$$e = 0, \quad \rho = a, \quad \phi = \nu t. \tag{197}$$

The differential equations become

$$\begin{cases} a^3 \dfrac{d^2X}{dt^2} + (1 - 3\cos^2\nu t)\,\lambda X - (3 \sin \nu t \cos \nu t)\,\lambda Y = 0, \\ a^3 \dfrac{d^2Y}{dt^2} + (3 \sin \nu t \cos \nu t)\,\lambda X + (1 - 3\sin^2 \nu t)\,\lambda Y = 0; \end{cases} \tag{198}$$

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or

$$(199)\qquad \begin{cases} a^2\dfrac{d^2X}{dt^2} - \dfrac{\lambda X}{2} = \dfrac{3\lambda}{2}(X\cos 2\nu t + Y\sin 2\nu t), \\ a^2\dfrac{d^2Y}{dt^2} - \dfrac{\lambda Y}{2} = \dfrac{3\lambda}{2}(X\sin 2\nu t - Y\cos 2\nu t), \end{cases}$$

These are known forms and their integrals are written down immediately, namely

$$(200)\qquad \begin{cases} X = e^{\alpha t}(P\cos\nu t - Q\sin\nu t), \\ Y = e^{\alpha t}(P\sin\nu t + Q\cos\nu t). \end{cases}$$

in which P and Q are determined by the equations

$$(201)\qquad \begin{cases} P\left(\alpha^2 - \nu^2 - \dfrac{2\lambda}{a^2}\right) - 2\alpha\nu Q = 0, \\ 2\alpha\nu P + \left(\alpha^2 - \nu^2 + \dfrac{\lambda}{a^2}\right) Q = 0, \end{cases}$$

These equations being linear and homogeneous in P, Q, their determinant must vanish; this gives an equation for α which has a pure imaginary root if

$$(202)\qquad 9\lambda > 8k^2 > 4\lambda.$$

Passing now to the general case in which the orbits are elliptical, in virtue of the relations

$$(203)\qquad \rho^2\frac{d\phi}{dt} = c, \quad \rho = \frac{p}{1+e\cos\phi}, \quad c = k\sqrt{p},$$

the differential equations become transformed by the substitutions

$$(204)\qquad X = \rho U, \quad Y = \rho V,$$

$$(205)\qquad U = P\cos\phi - Q\sin\phi, \quad V = P\sin\phi + Q\cos\phi,$$

into the forms

$$(206)\qquad \begin{cases} k^2(1+e\cos\phi)\left(\dfrac{d^2P}{d\phi^2} - \dfrac{2dQ}{d\phi}\right) - (k^2+2\lambda)P = 0, \\ k^2(1+e\cos\phi)\left(\dfrac{d^2Q}{d\phi^2} + \dfrac{2dP}{d\phi}\right) - (k^2-\lambda)\ Q = 0, \end{cases}$$

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The integration of these linear differential equations may be approached either by the means employed by Poincaré in the second volume of his *Méthodes Nouvelles* with reference to linear equations with periodic coefficients, or by resort to developments in series of powers of the eccentricity by putting

$$(207) \qquad P = P_0 + \Sigma i\, P_i e^i, \quad Q = Q_0 + \Sigma i\, Q_i e^i,$$

and comparing the coefficients of like powers of e. We obtain thus for P and Q systems of linear non-homogeneous differential equations with constant coefficients

$$(208) \quad \begin{cases} k^2\left(\dfrac{d^2P_0}{d\phi^2} - \dfrac{2dQ_0}{d\phi}\right) - (k^2 + 2\lambda)\, P_0 = 0, \\ k^2\left(\dfrac{d^2Q_0}{d\phi^2} + \dfrac{2dP_0}{d\phi}\right) - (k^2 - \lambda)\ \ Q_0 = 0, \\ k^2\left(\dfrac{d^2P_1}{d\phi^2} - \dfrac{2dQ_1}{d\phi}\right) - (k^2 + 2\lambda)\, P_1 = -(k^2+2\lambda)\, P_0 \cos\phi, \\ k^2\left(\dfrac{d^2Q_1}{d\phi^2} + \dfrac{2dP_1}{d\phi}\right) - (k^2 - \lambda)\ \ Q_1 = -(k^2 - \lambda)\, Q_0 \cos\phi, \\ \text{and so on,} \end{cases}$$

whose integration offers no difficulty.

ON THE COEFFICIENTS OF CAPACITY OF TWO SPHERES.

By E. W. Barnes.

§ 1. THE object of the present paper is to shew that the expressions for the coefficients of capacity of two non-intersecting electrified spheres,* in presence of one another,

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* It is convenient to refer to

a. Maxwell, *Magnetism and Electricity*, 3rd edition, Vol. I., pp. 266–283.
b. Poisson, *Mem. de l'Institut* (1811) (I.), pp. 1 et seq. (II)., 163 et seq.
c. Kirchoff, *Crelle* (1861), Vol. LIX., pp. 89 et seq.
d. Hicks, *Phil. Trans. Roy. Soc.* (1880). Vol. CLXXI., pp. 455 et seq.
e. Herman, *Quarterly Journ. of Mathematics*, Vol. XXII. (1887), pp. 204 et seq.

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can be reduced to functions which I have proposed to call double Gamma-functions.*

The functions which arise as coefficients of capacity are met with in other physical investigations connected with two spheres, and they are intimately connected with Lambert's series.† For this reason I have given the name Lambert's function to a certain uniform integral function, and as a conclusion to the present paper I have expressed Lambert's function in terms of double and simple Gamma-functions, and have obtained definite integrals analogous to those of Poisson to express its logarithmic derivates.

§ 2. Let A and B be the centres of the two spheres, C_1 and C_2 their common inverse points. Let O be the middle point of C_1C_2 and let $C_1C_2 = 2\lambda$.

From the dipolar transformation represented by

$$x + \iota y = \lambda \tan \tfrac{1}{2}(u + \iota v)$$

it proves convenient to take

$\alpha = \log \dfrac{C_2P}{C_1P}$, where P is any point on the sphere of centre A,

$\beta = \log \dfrac{C_1P}{C_2P}$, ... B,

so that α and β are always positive.

If a and b are the radii of the two spheres and c the distance between their centres, we readily find

$$a = \lambda/\sinh\alpha, \quad b = \lambda/\sinh\beta, \quad c = \lambda(\coth\alpha + \coth\beta),$$

and therefore $c = \dfrac{ab}{\lambda}\sinh\varpi$, where $\varpi = \alpha + \beta$.

The parameter λ is given in terms of a, b, and c by the relation

$$a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2 = 4c^2\lambda^2.$$

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* The theory has been developed by me in the papers :—

a. "The theory of the Gamma-function," *Messenger of Mathematics*, Vol. XXIX pp. 64–128.

b. "The theory of the G function," *Quarterly Journal of Mathematics* Vol. XXXI., pp. 264–314.

c. "The genesis of the double Gamma-functions," *Proc. Lond. Math. Soc.*, Vol. XXXI, pp. 358–381.

d. "The theory of the double Gamma-function," *Phil. Trans. Roy. Soc.* (A), (1901), Vol. CXCVI., pp. 265–387.

† For the connection between Lambert's series and elliptic functions, see Jacobi, *Fundamenta Nova*, § 66.

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With this notation, if the two spheres are maintained at potentials V_a and V_b by charges E_a and E_b, we have

$$E_a = q_{aa} V_a + q_{ab} V_b,$$

$$E_b = q_{ab} V_a + q_{bb} V_b,$$

and Maxwell shews by means of Lord Kelvin's system of images that

$$q_{aa} = \lambda \sum_{s=0}^{\infty} \frac{1}{\sinh(\alpha + s\varpi)},$$

$$q_{ab} = -\lambda \sum_{s=1}^{\infty} \frac{1}{\sinh s\varpi},$$

$$q_{bb} = \lambda \sum_{s=0}^{\infty} \frac{1}{\sinh(\beta + s\varpi)}.$$

Maxwell's notation has been slightly altered in order to secure that α, β, and ϖ are always *positive* quantities.

§ 3. We proceed now to consider the transformation of the series for q_{aa}/λ.

We have evidently

$$\frac{q_{aa}}{\lambda} = \sum_{s=0}^{\infty} \left[\frac{1}{\alpha + s\varpi} + \sum_{n=-\infty}^{\infty}{}' (-)^n \left\{ \frac{1}{\alpha + s\varpi + n\pi\iota} - \frac{1}{n\pi\iota} \right\} \right],$$

and the latter expression is evidently, in the limit when m is infinite, the same as

$$2 \sum_{s=0}^{pm} \left[\frac{1}{\alpha + s\varpi} + \sum_{n=-qm}^{qm}{}' \left\{ \frac{1}{\alpha + s\varpi + 2n\pi\iota} - \frac{1}{2n\pi\iota} \right\} \right]$$

$$- \sum_{s=0}^{pm} \left[\frac{1}{\alpha + s\varpi} + \sum_{n=-2qm}^{2qm}{}' \left\{ \frac{1}{\alpha + s\varpi + n\pi\iota} - \frac{1}{n\pi\iota} \right\} \right],$$

where p and q are integers, and $\frac{q}{p}$ is very large.

Let us seek then to transform the expression

$$f(\alpha, \varpi) = \sum_{s=0}^{pm} \left[\frac{1}{\alpha + s\varpi} + \sum_{n=-qm}^{qm}{}' \left\{ \frac{1}{\alpha + s\varpi + n\varpi'} - \frac{1}{n\varpi'} \right\} \right],$$

where m is a very large integer which is ultimately to be made infinite.

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§ 4. We have at once

$$f(\alpha, \varpi) = \sum_{s=0}^{pm} \left[\sum_{n=0}^{qm} \left\{ \frac{1}{\alpha + s\varpi + n\varpi'} - \frac{1}{s\varpi + n\varpi'} + \frac{\alpha}{(s\varpi + n\varpi')^2} \right\} \right.$$

$$+ \sum_{n=0}^{qm} \left\{ \frac{1}{\alpha + s\varpi - n\varpi'} - \frac{1}{s\varpi - n\varpi'} + \frac{\alpha}{(s\varpi - n\varpi')^2} \right\}$$

$$+ \sum_{n=-qm}^{qm}{}' \left\{ \frac{1}{s\varpi + n\varpi'} - \frac{1}{n\varpi'} \right\} - \sum_{n=-qm}^{qm}{}' \frac{\alpha}{(s\varpi + n\varpi')^2}$$

$$\left. - \frac{1}{\alpha + s\varpi} + \frac{2}{s\varpi} - \frac{2\alpha}{s^2\varpi^2} \right].$$

Now if $\Gamma_2(\alpha \mid \varpi, \varpi')$ be the double Gamma-function of α with parameters ϖ and ϖ', we have [D. G. F., p. 299]*

$$\Gamma_2^{-1}(\alpha) = e^{\gamma_{21}\frac{\alpha^2}{2!} + \gamma_{22}\alpha} . \alpha . \prod_0^\infty{}' \prod_{m_1 m_2} \left[\left(1 + \frac{\alpha}{\Omega}\right) e^{-\frac{\alpha}{\Omega} + \frac{\alpha^2}{2\Omega^2}} \right],$$

where $\Omega = m_1\varpi + m_2\varpi'$, and γ_{21} and γ_{22} are the first and second double Gamma-modular functions of ϖ and ϖ'.

Therefore

$$-\psi_2'(\alpha) = -\frac{d}{d\alpha} \log \Gamma_2(\alpha)$$

$$= \gamma_{21}\alpha + \gamma_{22} + \frac{1}{\alpha} + \sum_0^\infty{}' \left[\frac{1}{\alpha + \Omega} - \frac{1}{\Omega} + \frac{\alpha}{\Omega^2} \right].$$

The double series is convergent in a manner independent of the ratio of the infinite limits of m and m'.

Hence

$$f(\alpha, \varpi) = -\psi_2'(\alpha \mid \varpi, \varpi') - \psi_2'(\alpha \mid \varpi, -\varpi')$$

$$- \alpha[\gamma_{21}(\varpi, \varpi') + \gamma_{21}(\varpi, -\varpi')] - [\gamma_{22}(\varpi, \varpi') + \gamma_{22}(\varpi, -\varpi')]$$

$$- \sum_{s=0}^{pm} \frac{1}{\alpha + s\varpi} + \sum_{s=1}^{pm} \left(\frac{2}{s\varpi} - \frac{2\alpha}{s^2\varpi^2} \right)$$

$$+ \sum_{s=0}^{pm} \left[\sum_{n=-qm}^{qm}{}' \left(\frac{1}{s\varpi + n\varpi'} - \frac{1}{n\varpi'} \right) \right] - \sum_{s=0}^{pm} \left[\sum_{n=-qm}^{qm}{}' \frac{\alpha}{(s\varpi + n\varpi')^2} \right].$$

* The papers *a* and *d* cited in § 1 will be referred to throughout as G. F. and D. G. F. respectively.

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Now [G. F., p. 65],

$$\psi_1'(\alpha\,|\,\varpi) = -\frac{\gamma}{\varpi} + \frac{\log\varpi}{\varpi} - \frac{1}{\alpha} - \sum_{s=1}^{\infty}\left[\frac{1}{\alpha+s\varpi} - \frac{1}{s\varpi}\right],$$

and therefore

$$\begin{aligned} &f(\alpha,\varpi) + \psi_2'(\alpha\,|\,\varpi,\varpi') + \psi_2'(\alpha\,|\,\varpi,-\varpi') - \psi_1'(\alpha\,|\,\varpi) \\ &= -\alpha[\gamma_{21}(\varpi,\varpi') + \gamma_{21}(\varpi,-\varpi')] - [\gamma_{22}(\varpi,\varpi') + \gamma_{22}(\varpi,-\varpi')] \\ &\quad + \frac{\gamma}{\varpi} - \frac{\log\varpi}{\varpi} - \sum_{s=1}^{pm}\frac{1}{s\varpi} + \sum_{s=0}^{pm}\sum_{n=0}^{qm}{}'\left\{\frac{1}{s\varpi+n\varpi'} + \frac{1}{s\varpi-n\varpi'}\right\} \\ &\quad - \alpha\sum_{s=0}^{pm}\sum_{n=0}^{qm}{}'\left\{\frac{1}{(s\varpi+n\varpi')^2} + \frac{1}{(s\varpi-n\varpi')^2}\right\}. \end{aligned}$$

§ 5. Now it has been established in the theory of the double Gamma-function (D. G. F., p. 337) that

$$\begin{aligned} &\sum_{s=0}^{pm}\sum_{n=0}^{qm}{}'\frac{1}{(s\varpi+n\varpi')^2} + \gamma_{21}(\varpi,\varpi') \\ &\quad = -\frac{1}{\varpi\varpi'}\{\log(p\varpi+q\varpi') - \log p\varpi - \log q\varpi' - \log m\}, \end{aligned}$$

together with terms which vanish when m is infinite.

The principal values of the logarithms with respect to the axis of $-(\varpi+\varpi')$ are throughout to be taken.

Now in the case which we are considering, ϖ is purely real and positive, and $\varpi' = 2\pi\iota$ is purely imaginary. Thus the cross-cut with respect to which the logarithms are taken lies in the third quadrant. And therefore the imaginary part of $\log(p\varpi+q\varpi')$ lies between 0 and $\frac{\pi\iota}{2}$, and the imaginary part of $\log\varpi'$ is $\frac{\pi\iota}{2}$.

Similarly

$$\begin{aligned} &\sum_{s=0}^{pm}\sum_{n=0}^{qm}{}'\frac{1}{(s\varpi-n\varpi')^2} + \gamma_{21}(\varpi,-\varpi') \\ &\quad = \frac{1}{\varpi\varpi'}\{\log(p\varpi-q\varpi') - \log p\varpi - \log(-q\varpi') - \log m\}, \end{aligned}$$

the logarithms having their principal values with respect to an axis in the second quadrant. Thus the imaginary part of

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$\log(p\varpi - q\varpi')$ lies between 0 and $-\frac{\pi\iota}{2}$, and that of $\log(-q\varpi')$ is $-\frac{\pi\iota}{2}$.

Therefore in the limit when m is infinite

$$\gamma_{21}(\varpi, \varpi') + \gamma_{21}(\varpi, -\varpi') + \sum_{s=0}^{pm} \sum_{n=0}^{qm}{}' \left\{ \frac{1}{(s\varpi + n\varpi')^3} + \frac{1}{(s\varpi - n\varpi')^3} \right\}$$
$$= -\frac{1}{\varpi\varpi'} \left\{ \log \frac{p\varpi + q\varpi'}{p\varpi - q\varpi'}, -\pi\iota \right\},$$

the imaginary part of the logarithm lying between $\pm\frac{\iota\pi}{2}$.

§ 6. Again, from the same theory (D. G. F., p. 336), we have

$$\sum_{s=0}^{pm} \sum_{n=0}^{qm}{}' \frac{1}{(s\varpi + n\varpi')} - \gamma_{22}(\varpi, \varpi')$$
$$= \left(\frac{1}{2\varpi} + \frac{1}{2\varpi'}\right) \{\log m - \log(p\varpi + q\varpi') + \log p\varpi + \log q\varpi'\}$$
$$+ \left(\frac{pm+1}{\varpi'} + \frac{qm+1}{\varpi}\right) \log(p\varpi + q\varpi')$$
$$- \frac{pm+1}{\varpi'} \log p\varpi - \frac{qm+1}{\varpi} \log q\varpi',$$

+ terms which vanish when m is infinite. The principal values of the logarithms with respect to $-(\varpi + \varpi')$ are to be taken.

The numbers m and m' which enter into the general theory vanish in the present case since ϖ is real, and ϖ' purely imaginary.

Hence, as in § 5,

$$\sum_{s=0}^{pm} \sum_{n=0}^{qm}{}' \left\{ \frac{1}{s\varpi + n\varpi'} + \frac{1}{s\varpi - n\varpi'} \right\} - \gamma_{22}(\varpi, \varpi') - \gamma_{22}(\varpi, -\varpi')$$
$$= \frac{1}{\varpi} \log mp\varpi - \frac{2qm+1}{\varpi} \log q\varpi' + \pi\iota \left\{ \frac{2qm+1}{2\varpi} + \frac{1}{2\varpi'} \right\}$$
$$+ \left\{ \frac{2pm+1}{2\varpi'} + \frac{2qm+1}{2\varpi} \right\} \log(p\varpi + q\varpi')$$
$$+ \left\{ \frac{2qm+1}{2\varpi} - \frac{2pm+1}{2\varpi'} \right\} \log(p\varpi - q\varpi').$$

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Therefore, in the limit when m is infinite,

$$f(\alpha, \varpi) + \psi_2'(\alpha \mid \varpi, \varpi') + \psi_2'(\alpha \mid \varpi, -\varpi') - \psi_1'(\alpha \mid \varpi)$$
$$= \frac{\alpha}{\varpi\varpi'}\left\{\log\frac{p\varpi + q\varpi'}{p\varpi - q\varpi'} - \pi\iota\right\} - \frac{2qm+1}{\varpi}\log q\varpi'$$
$$+ \pi\iota\left\{\frac{2qm+1}{2\varpi} + \frac{1}{2\varpi'}\right\} + \frac{2pm+1}{2\varpi'}\log\frac{p\varpi + q\varpi'}{p\varpi - q\varpi'}$$
$$+ \frac{2qm+1}{2\varpi}\log[(p\varpi + q\varpi')(p\varpi - q\varpi')].$$

§ 7. Hence, in the limit when m is infinite, and $\frac{q}{p}$ is very large,

$$\frac{q_{\alpha\alpha}}{\lambda} = \psi_2'(\alpha \mid \varpi, \pi\iota) + \psi_2'(\alpha \mid \varpi, -\pi\iota) - \psi_1'(\alpha \mid \varpi)$$
$$- 2\psi_2'(\alpha \mid \varpi, 2\pi\iota) - 2\psi_2'(\alpha \mid \varpi, -2\pi\iota) + 2\psi_1'(\alpha \mid \varpi)$$
$$- 4\frac{2qm+1}{2\varpi}\log 2\pi\iota q + 2\pi\iota\left\{\frac{2qm+1}{2\varpi} + \frac{1}{4\pi\iota}\right\}$$
$$+ \frac{4qm+2}{2\varpi}\log 2\pi\iota q - \pi\iota\left\{\frac{4qm+1}{2\varpi} + \frac{1}{2\pi\iota}\right\}$$
$$+ \frac{2qm+1}{\varpi}\log(p^2\varpi^2 + 4\pi^2q^2) - \frac{4qm+1}{2\varpi}\log(p^2\varpi^2 + 4\pi^2q^2)$$
$$= \sum_{\varepsilon=\pm1}\{\psi_2'(\alpha \mid \varpi, \varepsilon\pi\iota) - 2\psi_2'(\alpha \mid \varpi, 2\varepsilon\pi\iota)\} + \psi_1'(\alpha \mid \varpi)$$
$$+ \frac{1}{2\varpi}\log(p^2\varpi^2 + 4\pi^2q^2) - \frac{1}{\varpi}\log 2\pi\iota q + \frac{\pi\iota}{2\varpi}.$$

The last three terms are equal to

$$\frac{1}{2\varpi}\log\left(1 + \frac{p^2\varpi^2}{4q^2\pi^2}\right),$$

and therefore their sum tends to zero as $\frac{q}{p}$ tends to infinity.

We have then, finally,

$$\frac{q_{\alpha\alpha}}{\lambda} = \sum_{\varepsilon=\pm1}\{\psi_2'(\alpha \mid \varpi, \varepsilon\pi\iota) - 2\psi_2'(\alpha \mid \varpi, 2\varepsilon\pi\iota)\} + \psi_1'(\alpha \mid \varpi),$$

which is the required expression for $\frac{q_{\alpha\alpha}}{\lambda}$ in terms of double Gamma-functions.

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§ 8. We find, similarly,

$$\frac{q_{ab}}{\lambda} = \sum_{\varepsilon=\pm 1} \{\psi_2'(\beta \mid \varpi, \varepsilon\pi\iota) - 2\psi_2'(\beta \mid \varpi, 2\varepsilon\pi\iota)\} + \psi_1'(\beta \mid \varpi),$$

and

$$-\frac{q_{aa}}{\lambda} = \sum_{\varepsilon=\pm 1} \{\psi_2'(\varpi \mid \varpi, \varepsilon\pi\iota) - 2\psi_2'(\varpi \mid \varpi, 2\varepsilon\pi\iota)\} + \psi_1'(\varpi \mid \varpi).$$

Now it has been proved [D. G. F., p. 288] that, in the limit when $a = 0$,

$$\psi_2'(\varpi \mid \varpi, \varpi') = \left[\psi_2'(a \mid \varpi, \varpi') + \frac{1}{a}\right] - \left[\psi_1'(a \mid \varpi') + \frac{1}{a}\right]$$

$$= -\gamma_{22}(\varpi, \varpi') + \frac{\gamma}{\varpi'} - \frac{\log \varpi'}{\varpi'}.$$

therefore

$$\frac{q_{ab}}{\lambda} = \sum_{\varepsilon=\pm 1} [\gamma_{22}(\varpi, \varepsilon\pi\iota) - 2\gamma_{22}(\varpi, 2\varepsilon\pi\iota)] + \frac{\gamma}{\varpi} - \frac{\log \varpi}{\varpi};$$

or, as we may, if we please, write it [G. F., p. 66],

$$\frac{q_{ab}}{\lambda} = \sum_{\varepsilon=\pm 1} [\gamma_{22}(\varpi, \varepsilon\pi\iota) - 2\gamma_{22}(\varpi, 2\varepsilon\pi\iota)] - \gamma_{11}(\varpi).$$

Thus q_{ab} is expressible in terms of Euler's constant and the second double Gamma-modular functions of ϖ and $\varepsilon\pi\iota$ where $\varepsilon = \pm 1$ or ± 2.

§ 9. It is possible to give a different form to the expressions for q_{aa} and q_{ab} by means of the formula of transformation of the parameters of the double Gamma-functions.

For, by this formula [D. G. F., p. 359],

$$\psi_2'\left(\alpha \mid \varpi, \frac{\varpi'}{2}\right) = \psi_2'(\alpha \mid \varpi, \varpi') + \psi_2'\left(\alpha + \frac{\varpi'}{2} \mid \varpi, \varpi'\right).$$

Hence

$$\frac{q_{aa}}{\lambda} = \sum_{\varepsilon=\pm 1} \{\psi_2'(\alpha + \varepsilon\pi\iota \mid \varpi, 2\varepsilon\pi\iota) - \psi_2'(\alpha \mid \varpi, 2\varepsilon\pi\iota)\} + \psi_1'(\alpha \mid \varpi),$$

in which the parameters of the double Gamma-functions are the same quantities ϖ, $2\varepsilon\pi\iota$.

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The corresponding formula for $-\frac{q_{\alpha\beta}}{\lambda}$ is at once obtained by making $\alpha = \varpi$.

It forms a useful verification of the expression for $\frac{q_{\alpha\alpha}}{\lambda}$ to show that, *quâ* function of α, it satisfies the difference equations

$$f(\alpha + \varpi) = f(\alpha) - \operatorname{cosech} \alpha,$$
$$f(\alpha + 2\pi\iota) = f(\alpha).$$

The expression for the kinetic energy of two spheres moving in an infinite fluid along the line of centres depends* on series of the type

$$\sum_{s=1}^{\infty} \frac{1}{\sinh^3(\alpha + s\varpi)}.$$

Since $\frac{d^2}{d\alpha^2} \frac{1}{\sinh(\alpha + s\varpi)} = \frac{2}{\sinh^3(\alpha + s\varpi)} + \frac{1}{\sinh(\alpha + s\varpi)}$,

we see that it can be conveniently expressed it terms of the functions

$$\psi_2'(\alpha \mid \varpi, \pm 2\pi\iota) \text{ and } \psi_2^{(2)}(\alpha \mid \varpi, \pm 2\pi\iota).$$

§ 10. We have hitherto assumed that α, β, and ϖ are the real positive quantities which arise in Maxwell's investigation. We proceed next to consider the case when α is any complex variable. We shall still assume ϖ to be real in order to avoid the complexity which arises when the axes of ϖ and $s\pi\iota$ include the axis to the point -1.

We shall construct what it is proposed to call Lambert's function of α, with parameter ϖ.

The function is given by

$$L(\alpha \mid \varpi) = \prod_{s=0}^{\infty} \{1 - e^{-\alpha - s\varpi}\}.$$

It is obviously an integral function with zeros at the points

$$\alpha = -s\varpi + 2k\pi\iota, \quad \begin{cases} k = -\infty \ldots -2, -1, 0, 1, 2, \ldots, \infty, \\ s = 0, 1, 2, \ldots, \infty; \end{cases}$$

and it is periodic with period $2\pi\iota$.

It thus lies in complexity between the Gamma and double Gamma functions. It will be found possible to express it in terms of these functions, and the modular constants which arise will be derived from sums of the type

$$\Sigma\gamma_{s_1}(\varpi, \pm 2\pi\iota), \quad \Sigma\gamma_{s_2}(\varpi, \pm 2\pi\iota), \quad \Sigma \log \rho_s(\varpi, \pm 2\pi\iota),$$

* Herman (loc. cit. § 1), p. 216.

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and it will be therefore possible to express such combinations in terms of the series which may be derived from Lambert's function.

The asymptotic expansion of the logarithm of the function derived from Lambert's function by the substitution $-e^{-\alpha}=z$, has, when $|z|$ is large, been previously obtained by me.* It should be noticed, however, that this asymptotic expansion fails to give the asymptotic expansion for $\log L(\alpha)$ as the transformation $z=-e^{-\alpha}$, introduces a set of zeros which ultimately cover the α-plane when $\mathfrak{R}(\alpha)$ is negative and $|\alpha|$ is very large.

§ 11. From the definition of Lambert's function, we have

$$L(\alpha+2\pi\iota)=L(\alpha),$$

$$L(\alpha+\varpi)=\frac{L(\alpha)}{1-e^{-\alpha}}.$$

Consider now

$$F(\alpha)=\frac{\Gamma_2^{-1}(\alpha\,|\,\varpi,2\pi\iota)\,\Gamma_2^{-1}(\alpha\,|\,\varpi,-2\pi\iota)}{\Gamma_1^{-1}(\alpha\,|\,\varpi)}.$$

It is an integral function with zeros of the same type at the same points as $L(\alpha)$.

Also [D. G. F., p. 293]

$$\frac{F(\alpha+\varpi)}{F(\alpha)}=\frac{\Gamma_1(\alpha\,|\,2\pi\iota)\,\Gamma_1(\alpha\,|\,-2\pi\iota)}{1/\alpha},$$

for, if ϖ be purely real, the numbers m and m' of the general theory vanish.

Now
$$\frac{1}{\alpha}\Gamma^{-1}(\alpha\,|\,2\pi\iota)\,\Gamma_1^{-1}(\alpha\,|\,-2\pi\iota)$$
$$=\alpha\prod_{n=1}^{\infty}\left[\left\{\left(1+\frac{\alpha}{2n\pi\iota}\right)e^{-\frac{\alpha}{2n\pi\iota}}\right\}\left\{\left(1-\frac{\alpha}{2n\pi\iota}\right)e^{\frac{\alpha}{2n\pi\iota}}\right\}\right]$$
$$\times\exp.\left\{\frac{\gamma}{2\pi\iota}-\frac{\log(2\pi\iota)}{2\pi\iota}-\frac{\gamma}{2\pi\iota}+\frac{\log(-2\pi\iota)}{2\pi\iota}\right\}\alpha$$
$$=\alpha\prod_{n=1}^{\infty}\left\{1+\frac{\alpha^2}{(2n\pi)^2}\right\}e^{-\frac{\alpha}{2}}=\frac{2\sin\frac{\alpha\iota}{2}}{\iota}.e^{-\frac{\alpha}{2}}$$
$$=1-e^{-\alpha}.$$

* 'A memoir on integral functions,' *Phil. Trans. Roy. Soc.* (A), Vol. CXCIX., pp. 466 and 467.

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Thus $$F(\alpha+\varpi)=\frac{F(\alpha)}{1-e^{-\alpha}}.$$

Again [D. G. F., p. 293]

$$\frac{F(\alpha+2\pi\iota)}{F(\alpha)}=\frac{\Gamma_1(\alpha\,|\,\varpi)}{\sqrt{\left(\frac{2\pi}{\varpi}\right)}}\cdot\frac{\sqrt{\left(\frac{2\pi}{\varpi}\right)}}{\Gamma_1(\alpha+2\pi\iota\,|\,\varpi)}\cdot\frac{\Gamma_1(\alpha+2\pi\iota\,|\,\varpi)}{\Gamma_1(\alpha\,|\,\varpi)}$$

$$=1.$$

Hence $\frac{L(\alpha\,|\,\varpi)}{F(\alpha)}$ is a function with no poles which is doubly periodic with periods ϖ and $2\varpi\iota$; it is therefore a constant.

Hence $$L(\alpha\,|\,\varpi)=C\frac{\Gamma_2^{-1}(\alpha\,|\,\varpi,\,2\pi\iota)\,\Gamma_2^{-1}(\alpha\,|\,\varpi,\,-2\pi\iota)}{\Gamma_1^{-1}(\alpha\,|\,\varpi)}.$$

To determine the constant C, let α be large, real, and positive. Then $L(\alpha\,|\,\varpi)=1$, and from the asymptotic expansion for the double Gamma-function (p. 379) we have

$$\log\Gamma_2^{-1}(\alpha)=-\log\rho_2(\varpi,2\pi\iota)+{}_2S_1'(\alpha)\log\alpha-\alpha\,{}_2S_1^{(2)}(0)-\frac{3\alpha^2}{4}\,{}_2S_1^{(3)}(0)$$

$$+\text{ terms which vanish with } 1/\alpha.$$

Hence

$$0=\log C-\log\rho_2(\varpi,2\pi\iota)-\log\rho_2(\varpi,-2\pi\iota)+\tfrac{1}{2}\log\frac{2\pi}{\varpi},$$

Thus $$C=\frac{\rho_2(\varpi,2\pi\iota)\,\rho_2(\varpi,-2\pi\iota)}{\rho_1(\varpi)},$$

putting $\rho_1(\varpi)=\sqrt{\left(\frac{2\pi}{\varpi}\right)}$ [as in D. G. F., p. 300].

We have now obtained the expression of Lambert's function in terms of simple and double Gamma-functions. It is evidently possible to extend the previous investigation to the case when ϖ is any complex quantity (it is valid as it stands when $\Re(\varpi)$ is positive) should it be thought worth while to introduce the numbers m and m' of the general theory.

§ 12. From the result of the preceding paragraph we may again deduce the expression for $\frac{q_{aa}}{\lambda}$ given in § 9. The

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expression is valid for all complex values of α, and all complex values of ϖ for which $\mathfrak{R}(\varpi)$ is positive.

For we have

$$L(\alpha) = \prod_{s=0}^{\infty} [1 - e^{-\alpha - s\varpi}],$$

and therefore

$$L(\alpha + \pi\iota) = \prod_{s=0}^{\infty} [1 + e^{-\alpha - s\varpi}].$$

Let us now take $l'(\alpha) = \dfrac{d}{d\alpha} \log L(\alpha)$, then we have

$$l'(\alpha) - l'(\alpha + \pi\iota) = \sum_{s=0}^{\infty} \left\{ \frac{1}{e^{\alpha+s\varpi} - 1} + \frac{1}{e^{\alpha+s\varpi} + 1} \right\}$$

$$= \sum_{s=0}^{\infty} \operatorname{cosech}(\alpha + s\varpi) = \frac{q_{\alpha\varpi}}{\lambda}.$$

Hence

$$\frac{q_{\alpha\varpi}}{\lambda} = \psi_2'(\alpha + \pi\iota \,|\, \varpi, 2\pi\iota) - \psi_2'(\alpha \,|\, \varpi, 2\pi\iota)$$
$$+ \psi_2'(\alpha + \pi\iota \,|\, \varpi, -2\pi\iota) - \psi_2'(\alpha \,|\, \varpi, -2\pi\iota)$$
$$+ \psi_1'(\alpha \,|\, \varpi) - \psi_1'(\alpha + \pi\iota \,|\, \varpi).$$

Now [D. G. F., p. 291]

$$\psi_2'(\alpha - \pi\iota \,|\, \varpi, -2\pi\iota) = \psi_2'(\alpha + \pi\iota \,|\, \varpi, -2\pi\iota) - \psi_1'(\alpha + \pi\iota \,|\, \varpi).$$

Hence

$$\frac{q_{\alpha\varpi}}{\lambda} = \sum_{\epsilon=\pm 1} \{\psi_2'(\alpha + \epsilon\pi\iota \,|\, \varpi, 2\epsilon\pi\iota) - \psi_2'(\alpha \,|\, \varpi, 2\epsilon\pi\iota)\} + \psi_1'(\alpha \,|\, \varpi).$$

which agrees with the previous result.

§ 13. We can now prove that, when $\mathfrak{R}(\varpi)$ is positive,

$$\rho_2(\varpi, 2\pi\iota)\, \rho_2(\varpi, -2\pi\iota) = \rho_1(\varpi) \prod_{s=1}^{\infty} \{1 - e^{-s\varpi}\}.$$

For we have

$$L(\alpha) = C \frac{\Gamma_2^{-1}(\alpha \,|\, \varpi, 2\pi\iota)\, \Gamma_2^{-1}(\alpha \,|\, \varpi, -2\pi\iota)}{\Gamma_1^{-1}(\alpha, \varpi)}.$$

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Make now α tend to zero, then since

$$\underset{\alpha=0}{\text{Lt.}}\, \alpha\Gamma_2(\alpha) = 1, \quad \underset{\alpha=0}{\text{Lt.}}\, \alpha\Gamma_1(\alpha) = 1, \quad \underset{\alpha}{\text{Lt.}}\, \frac{L(\alpha)}{\alpha} = \prod_{s=1}^{\infty}(1 - e^{-s\varpi}),$$

we have $$C = \prod_{s=1}^{\infty}\{1 - e^{-s\varpi}\}.$$

We therefore have the identity stated.

As a corollary we notice that

$$\frac{\rho_2(\varpi, 2\pi\iota)\,\rho_2(\varpi, -2\pi\iota)}{\rho_1(\varpi)}$$

is periodic in ϖ with period $2\pi\iota$.

§ 14. We may next shew that

$$-\Sigma\gamma_{22}(\varpi, \pm 2\pi\iota) = \gamma_{11}(\varpi) + \tfrac{1}{2} + \sum_{s=1}^{\infty}\frac{1}{1 - e^{s\varpi}}.$$

For we have

$$l'(\alpha) = \psi_1'(\alpha \mid \varpi) - \Sigma\psi_2'(\alpha \mid \varpi, \pm 2\pi\iota) \ldots\ldots\ldots(1).$$

Now [D. G. F., p. 288]

$$\underset{\alpha=0}{\text{Lt.}}\left[\psi_2'(\alpha) + \frac{1}{\alpha}\right] = -\gamma_{22}(\varpi, 2\pi\iota),$$

and [G. F., p. 65]

$$\underset{\alpha=0}{\text{Lt.}}\left[\psi_1'(\alpha) + \frac{1}{\alpha}\right] = \gamma_{11}(\varpi).$$

Hence

$$\underset{\alpha=0}{\text{Lt.}}\left[l'(\alpha) - \frac{1}{\alpha}\right] = \gamma_{11}(\varpi) + \Sigma\gamma_{22}(\varpi, \pm 2\pi\iota).$$

But $$\underset{\alpha=0}{\text{Lt.}}\left[l'(\alpha) - \frac{1}{\alpha}\right] = \underset{\alpha=0}{\text{Lt.}}\left[\sum_{s=0}^{\infty}\left(\frac{1}{e^{\alpha+s\varpi} - 1}\right) - \frac{1}{\alpha}\right]$$

$$= -\tfrac{1}{2} + \sum_{s=1}^{\infty}\frac{1}{e^{s\varpi} - 1}.$$

Finally therefore

$$-\Sigma\gamma_{22}(\varpi, \pm 2\pi\iota) = \gamma_{11}(\varpi) + \tfrac{1}{2} + \sum_{s=1}^{\infty}\frac{1}{1 - e^{s\varpi}}.$$

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§ 15. We may similarly express in the form of a series

$$\Sigma\gamma_{s1}(\varpi, \pm 2\pi\iota).$$

For we have, on differentiating the equality (1) of the previous paragraph,

$$l^{(2)}(\alpha) = \psi_1^{(2)}(\alpha \mid \varpi) - \Sigma\psi_2^{(2)}(\alpha \mid \varpi, \pm 2\pi\iota).$$

Now [D. G. F., p. 287]

$$\operatorname*{Lt}_{\alpha=0}\left[\psi_2^{(2)}(\alpha \mid \varpi, 2\pi\iota) - \frac{1}{\alpha^2}\right] = -\gamma_{21}(\varpi, 2\pi\iota),$$

and

$$\operatorname*{Lt}_{\alpha=0}\left[\psi_1^{(2)}(\alpha \mid \varpi) - \frac{1}{\alpha^2}\right] = \frac{\pi^2}{6\varpi^2}.$$

Therefore, on making α tend to zero, we have

$$\operatorname*{Lt}_{\alpha=0}\left[l^{(2)}(\alpha) + \frac{1}{\alpha^2}\right] = \frac{\pi^2}{6\varpi^2} - \Sigma\gamma_{21}(\varpi, \pm 2\pi\iota).$$

But

$$\begin{aligned}\operatorname*{Lt}_{\alpha=0}&\left[l^{(2)}(\alpha) + \frac{1}{\alpha^2}\right]\\ &= \operatorname*{Lt}_{\alpha=0}\left\{-\sum_{s=0}^{\infty}\frac{e^{\alpha+s\varpi}}{(e^{\alpha+s\varpi}-1)^2} + \frac{1}{\alpha^2}\right\}\\ &= -\sum_{s=1}^{\infty}\frac{e^{s\varpi}}{(e^{s\varpi}-1)^2} + \operatorname*{Lt}_{\alpha=0}\left[\frac{1}{\alpha^2} - \frac{e^{\alpha}}{(e^{\alpha}-1)^2}\right]\\ &= \frac{1}{12} - \sum_{s=1}^{\infty}\frac{1}{4\sinh^2\frac{s\varpi}{2}}.\end{aligned}$$

Therefore

$$\Sigma\gamma_{21}(\varpi, \pm 2\pi\iota) = \frac{\pi^2}{6\varpi^2} - \frac{1}{12} + \sum_{s=1}^{\infty}\frac{1}{4\sinh^2\frac{s\varpi}{2}}.$$

§ 16. We proceed next to examine the degeneration of the line-integral expressions for logarithmic derivates of the double and simple Gamma-functions in the case of the combinations which form logarithmic derivates of Lambert's function.

We shall for simplicity assume that ϖ is real and positive, and, in order that the integrals employed may be convergent, that arg. (α) lies between $\pm\frac{\pi}{4}$.

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If L_1 denote the bisector of the fourth quadrant, *i.e.* of the right angle between the positive direction of the real axis and the negative direction of the imaginary axis, we have [D. G. F., p. 329], since the numbers M, m, and m' all vanish,

$$\psi_2'(\alpha|\varpi, 2\pi\iota) = -\int_0^\infty (L_1)\frac{dz}{z}\left\{\frac{ze^{-\alpha z}}{(1-e^{-\varpi z})(1-e^{-2\pi\iota z})}\right.$$
$$\left. - {}_2\frac{S_0^{(1)}(\alpha|\varpi, 2\pi\iota)}{z} + e^{-z}{}_2S_0'(\alpha|\varpi, 2\pi\iota)\right\}.$$

Hence [D. G. F., p. 273]

$$\Gamma(\alpha) = \psi_1'(\alpha|\varpi) - \psi_2'(\alpha|\varpi, 2\pi\iota) - \psi_2'(\alpha|\varpi, -2\pi\iota)$$
$$= \int_0^\infty (L_1)\frac{dz}{z}\left\{\frac{ze^{-\alpha z}}{(1-e^{-\varpi z})(1-e^{-2\pi\iota z})} - \frac{1}{\varpi . 2\pi\iota z} + e^{-z}{}_2S_0'(\alpha|\varpi, 2\pi\iota)\right\}$$
$$+ \int_0^\infty (L_2)\frac{dz}{z}\left\{\frac{ze^{-\alpha z}}{(1-e^{-\varpi z})(1-e^{2\pi\iota z})} + \frac{1}{\varpi . 2\pi\iota z} + e^{-z}{}_2S_0'(\alpha|\varpi, -2\pi\iota)\right\}$$
$$+ \psi_1'(\alpha|\varpi),$$

where L_2 bisects the first quadrant.

Consider now the integral along the line L_2. The subject of integration is one-valued, and is regular at the origin. If the integral be taken round a closed contour consisting of the positive direction of the real axis from the points 1, 2, 3, ... are excluded by small semi-circles *above* the real axis, an infinite circular arc, and the line L_2, it will by Cauchy's theorem vanish. The integral along the infinite circular arc obviously vanishes if the arc meets the real axis at some point between the series 1, 2, 3, ...

Hence

$$\int_0^\infty (L_2)\frac{dz}{z}\left\{\frac{ze^{-\alpha z}}{(1-e^{-\varpi z})(1-e^{2\pi\iota z})} + \frac{1}{\varpi . 2\pi\iota z} + e^{-z}{}_2S_0'(\alpha|\varpi, -2\pi\iota)\right\}$$
$$= \int_0^{1-\epsilon} + \int_{1+\epsilon}^{2-\epsilon} + \int_{2+\epsilon}^{3-\epsilon} + \ldots,$$

together with the sum of the integrals round the small semi-circular arcs

$$= P\int_0^\infty \frac{dz}{z}\left\{\frac{ze^{-\alpha z}}{(1-e^{-\varpi z})(1-e^{2\pi\iota z})} + \frac{1}{\varpi . 2\pi\iota z} + e^{-z}{}_2S_0'(\alpha|\varpi, -2\pi\iota)\right\}$$
$$+ \tfrac{1}{2}\sum_{n=1}^{\infty}\frac{e^{-\alpha n}}{1-e^{-\varpi n}},$$

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where the integral is taken along the real axis, and P denotes its principal value.

Proceed similarly with the integral for $\psi_2'(\alpha\,|\,\varpi, 2\pi\iota)$, and we find

$$l'(\alpha) = \psi_1'(\alpha\,|\,\varpi) + \sum_{n=1}^{\infty} \frac{e^{-\alpha n}}{1-e^{-\varpi n}} + P\int_0^\infty \frac{dz}{z}\left\{\frac{ze^{-\alpha z}}{1-e^{-\varpi z}}\left[\frac{1}{1-e^{2\pi\iota z}} + \frac{1}{1-e^{-2\pi\iota z}}\right] + e^{-z}\left[{}_2S_0'(\alpha\,|\,\varpi, -2\pi\iota) + {}_2S_0'(\alpha\,|\,\varpi, 2\pi\iota)\right]\right\}.$$

Now (D. G. F., p. 273),

$${}_2S_0'(\alpha\,|\,\varpi, \varpi') = \frac{\alpha}{\varpi\varpi'} - \left(\frac{1}{2\varpi} + \frac{1}{2\varpi'}\right).$$

Hence

$$l'(\alpha) = \sum_{n=1}^{\infty} \frac{e^{-\alpha n}}{1-e^{-\varpi n}} + \psi_1'(\alpha\,|\,\varpi) + \int_0^\infty \left\{\frac{ze^{-\alpha z}}{1-e^{-\varpi z}} - \frac{e^{-z}}{\varpi}\right\}\frac{dz}{z}.$$

But [G. F., p. 105]

$$\psi_1'(\alpha\,|\,\varpi) = \int_0^\infty \left\{\frac{-ze^{-\alpha z}}{1-e^{-\varpi z}} + \frac{e^{-z}}{\varpi}\right\}\frac{dz}{z},$$

and we are therefore left with

$$l'(\alpha) = \sum_{n=1}^{\infty} \frac{e^{-\alpha n}}{1-e^{-\varpi n}} = \sum_{n=1}^{\infty}\sum_{s=0}^{\infty} e^{-n(\alpha+s\varpi)} = \sum_{s=0}^{\infty} \frac{1}{e^{\alpha+s\varpi}-1}.$$

We thus return to the expression for $l'(\alpha)$ from which we started. Thus the integrals for the logarithmic derivates of the Gamma-functions verify the result of § 12, but fail to give definite integral expressions for logarithmic derivates of $L(\alpha\,|\,\varpi)$.

As a corallary notice that we have shewn that

$$l'(\alpha) = \int \frac{e^{-\alpha z}\,dz}{(1-e^{-\varpi z})(1-e^{-2\pi\iota z})}$$

taken round a contour which encloses the point $z=1$ but not the origin, and embraces the real axis.

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We can similarly show that

$$\frac{q_{aa}}{\lambda} = \int \frac{e^{-az}\,dz}{(1-e^{-\varpi z})(1+e^{-\pi iz})},$$

and

$$\frac{q_{ab}}{\lambda} = \int \frac{dz}{(1-e^{\varpi z})(1+e^{-\pi iz})},$$

the integrals being taken along the same contour.

§ 17. We have seen that the line integrals for $\log \Gamma_2(\alpha)$ and its derivates prove ineffective for $\log L(\alpha)$. We may, however, obtain, for the logarithmic derivates of Lambert's function, integrals analogous to those originally given by Poisson for the function q_{aa}/λ, and valid when $\mathfrak{R}(\alpha)$ is positive, ϖ being throughout assumed to be a real positive quantity.

If θ be real, we know by a result given by Legendre that

$$\int_0^\infty \frac{\sin \theta t}{e^{2\pi t}-1}\,dt = \tfrac{1}{4}\left\{\coth\frac{\theta}{2} - \frac{2}{\theta}\right\}$$

$$= \tfrac{1}{4}\left\{\frac{2}{1-e^{-\theta}} - 1 - \frac{2}{\theta}\right\}.$$

Hence, if $\mathfrak{R}(\alpha)$ be positive and ϖ be real,

$$\sum_{n=1}^{\infty}\int_0^\infty \frac{e^{-n\alpha}\sin n\varpi t}{e^{2\pi t}-1}\,dt = \sum_{n=1}^{\infty}\left\{-\frac{e^{-n\alpha}}{4} + \tfrac{1}{2}\frac{e^{-n\alpha}}{1-e^{-n\varpi}} - \frac{e^{-n\alpha}}{2n\varpi}\right\}.$$

Now

$$\frac{\sin\theta}{2(\cosh\alpha - \cos\theta)} = \sum_{n=1}^{\infty} e^{-n\alpha}\sin n\theta,$$

and

$$l'(\alpha) = \sum_{n=1}^{\infty}\frac{e^{-n\alpha}}{1-e^{-n\varpi}}.$$

Therefore

$$l'(\alpha) = \int_0^\infty \frac{\sin \varpi t}{\cosh\alpha - \cos\varpi t}\cdot\frac{dt}{e^{2\pi t}-1} - \frac{1}{2(1-e^{\alpha})} - \frac{1}{\varpi}\log(1-e^{-\alpha}).$$

We may write the last two terms in the form of a definite integral if we please.

For

$$\int_0^\infty \frac{\sin\varpi t}{t}\,\frac{dt}{\cosh\alpha - \cos\varpi t} = 2\int_0^\infty \sum_{n=1}^{\infty} e^{-n\alpha}\sin\varpi t\,\frac{dt}{t}$$

$$= \pi\sum_{n=1}^{\infty} e^{-n\alpha} = \frac{\pi}{e^{\alpha}-1}.$$

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And $$-\frac{1}{\varpi}\log(1-e^{-\alpha}) = \sum_{n=1}^{\infty}\frac{e^{-n\alpha}}{\varpi n} = \frac{1}{\varpi}\sum_{n=1}^{\infty}\int_0^{\infty} e^{-n(\alpha+x)}dx$$

$$= \int_0^{\infty}\frac{dt}{e^{\alpha+\varpi t}-1}.$$

Thus, finally,

$$l'(\alpha) = \int_0^{\infty}\left\{\frac{\sin\varpi t}{\cosh\alpha-\cos\varpi t}\left(\frac{1}{e^{2\pi t}-1}+\frac{1}{2\pi t}\right)+\frac{1}{e^{\alpha+\varpi t}-1}\right\}dt.$$

§ 18. The integral expression for q_{aa}/λ, valid when $\Re(\alpha)$ is positive, follows immediately.

For $$q_{aa}/\lambda = l'(\alpha) - l'(\alpha+\pi i),$$

and therefore

$$\frac{q_{aa}}{\lambda} = \int_0^{\infty}\left\{\frac{2\cosh\alpha\sin\varpi t}{\cosh^2\alpha-\cos^2\varpi t}\left(\frac{1}{e^{2\pi t}-1}+\frac{1}{2\pi t}\right)+\frac{2}{e^{\alpha+\varpi t}-e^{-(\alpha+\varpi t)}}\right\}dt$$

$$= \int_0^{\infty}\left\{\frac{4\cosh\alpha\sin\varpi t}{\cosh 2\alpha-\cos 2\varpi t}\left(\frac{1}{e^{2\pi t}-1}+\frac{1}{2\pi t}\right)+\frac{1}{\sinh(\alpha+\varpi t)}\right\}dt;$$

or as we may write it, if we please,

$$\frac{q_{aa}}{\lambda} = 4\cosh\alpha\int_0^{\infty}\frac{\sin\varpi t}{\cosh 2\alpha-\cos 2\varpi t}\cdot\frac{dt}{e^{2\pi t}-1}$$

$$+\frac{1}{2\sinh\alpha}-\frac{1}{\varpi}\log\tanh\frac{\alpha}{2}.$$

If now we make $\alpha=\varpi$, we obtain

$$-\frac{q_{ab}}{\lambda} = \int_0^{\infty}\left\{\frac{4\cosh\varpi\sin\varpi t}{\cosh 2\varpi-\cos 2\varpi t}\left(\frac{1}{e^{2\pi t}-1}+\frac{1}{2\pi t}\right)+\frac{1}{\sinh\varpi(t+1)}\right\}dt,$$

a form which is valid if ϖ be real and positive.

In connection with the first of these results, it may be observed that Poisson's formula is equivalent to

$$\frac{q_{aa}}{\lambda} = \frac{e^{-\alpha}}{1-e^{-\varpi}}+\int_0^{\infty}\frac{e^{-\alpha(t+1)}}{1-e^{-\varpi(t+1)}}dt$$

$$+2e^{-\alpha}\int_0^{\infty}\frac{e^{\varpi}\sin 2\alpha t-\sin 2(\alpha-\varpi)t}{(e^{2\pi t}-1)(\cosh\varpi-\cos 2\varpi t)}dt.$$

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§ 19. We may now write $\Sigma\gamma_{22}(\varpi, \pm 2\pi\iota)$ as a definite integral.

For by § 14 we have

$$-\Sigma\gamma_{22}(\varpi, \pm 2\pi\iota) = \gamma_{11}(\varpi) + \tfrac{1}{2} + \sum_{s=1}^{\infty} \frac{1}{1 - e^{s\varpi}} = \gamma_{11}(\varpi) + \tfrac{1}{2} - l'(\varpi \mid \varpi)$$

$$= -\int_0^\infty \frac{\sin \varpi t}{\cosh \varpi - \cos \varpi t} \cdot \frac{dt}{e^{2\pi t} - 1} + \frac{1}{2(1 - e^{\varpi})} + \frac{1}{\varpi} \log(1 - e^{-\varpi})$$

$$- \frac{\gamma}{\varpi} + \frac{\log \varpi}{\varpi} + \tfrac{1}{2}.$$

Now $$\gamma_{11}(\varpi) = \int_0^\infty \left\{ \frac{e^{-t}}{\varpi t} - \frac{1}{e^{\varpi t} - 1} \right\} dt,$$

and therefore we have

$$-\Sigma\gamma_{22}(\varpi, \pm 2\pi\iota)$$

$$= \int_0^\infty \left\{ \frac{-\sin \varpi t}{\cosh \varpi - \cos \varpi t} \left(\frac{1}{e^{2\pi t} - 1} + \frac{1}{2\pi t} \right) + \frac{e^{-t}}{\varpi t} - \frac{1}{e^{\varpi t} - 1} + \frac{1}{1 - e^{\varpi(t+1)}} \right\} dt + \tfrac{1}{2}.$$

§ 20. We may similarly obtain a definite integral for $\Sigma\gamma_{21}(\varpi, \pm 2\pi\iota)$. For by § 15 we have

$$\Sigma\gamma_{21}(\varpi, \pm 2\pi\iota) = \frac{\pi^2}{6\varpi^2} - \tfrac{1}{12} + \sum_{s=1}^{\infty} \frac{1}{4 \sinh^2 s\varpi/2}$$

$$= \frac{\pi^2}{6\varpi^2} - \tfrac{1}{12} - l^{(2)}(\varpi \mid \varpi).$$

Hence

$$\Sigma\gamma_{21}(\varpi, \pm 2\pi\iota)$$

$$= \frac{\pi^2}{6\varpi^2} - \tfrac{1}{12} + \int_0^\infty \left[\frac{\sinh \varpi \sin \varpi t}{(\cosh \varpi - \cos \varpi t)^2} \left\{ \frac{1}{e^{2\pi t} - 1} + \frac{1}{2\pi t} \right\} + \frac{e^{\varpi(t+1)}}{\{e^{\varpi(t+1)} - 1\}^2} \right] dt,$$

which is the expression required.

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We will not consider further the investigation of integrals of Poisson's type connected with logarithmic derivates of Lambert's function. Such a theory is obviously devoid of the elegance of the theory of the integrals in terms of which double Gamma-functions may be expressed. Further investigations must be devoted to the discovery of integrals of a different type in terms of which $l(\alpha|\varpi)$ and its derivates may be expressed.

§ 21. When $|\alpha|$ is large and $\mathfrak{R}(\alpha)$ is positive, we have an area in which there are no zeros of $L(\alpha|\varpi)$. Hence,* within this area there exists an asymptotic expansion for $\log L(\alpha)$. From the asymptotic expansions of the simple and double Gamma-functions it is interesting to verify that this expansion is zero of lower order than that of any negative power of α.

For, by § 11, we have

$$\log L(\alpha) = \sum_{\epsilon=\pm 1}\left[{}_2S_1'(\alpha|\varpi, 2\pi\iota\epsilon)\log\alpha - \frac{3\alpha^2}{4}{}_2S_1^{(3)}(0|\varpi, 2\pi\iota\epsilon) - \alpha\,{}_2S_1^{(2)}(0|\varpi, 2\pi\iota\epsilon) + \sum_{m=1}^{\infty}\frac{(-)^{m-1}\,{}_2S'_{m+1}(0|\varpi, 2\pi\iota\epsilon)}{m(m+1)\alpha^m}\right] + \left(\frac{\alpha}{\varpi}-\tfrac{1}{2}\right)\log\alpha - \frac{\alpha}{\varpi} + \sum_{m=1}^{\infty}\frac{(-)^{m-1}\,{}_1S'_{m+1}(0|\varpi)}{m(m+1)\alpha^m}.$$

Therefore

$$\log L(\alpha) = \sum_{\pm\iota}\left[\left\{\frac{\alpha^2}{4\varpi\pi\iota} - \alpha\left(\frac{1}{2\varpi}+\frac{1}{4\pi\iota}\right) + \frac{\varpi^2-4\pi^2+6\varpi\pi\iota}{24\pi\varpi\iota}\right\}\log\alpha + \alpha\left(\frac{1}{2\varpi}+\frac{1}{4\pi\iota}\right) - \frac{3\alpha^2}{8\pi\varpi\iota}\right] + \left(\frac{\alpha}{\varpi}-\tfrac{1}{2}\right)\log\alpha - \frac{\alpha}{\varpi} + \sum_{m=1}^{\infty}\frac{(-)^{m-1}}{m(m+1)\alpha^m}\{{}_2S'_{m+1}(0|\varpi, 2\pi\iota) + {}_2S'_{m+1}(0|\varpi, -2\pi\iota) + {}_1S'_{m+1}(0|\varpi)\}.$$

All the terms except those in the final series cancel with one another.

Again [D. G. F., p. 281, G. F., p. 84]

$$\frac{(-)^m}{(m+1)!}\{\Sigma\,{}_2S'_{m+1}(\alpha|\varpi, \pm 2\pi\iota) + {}_1S'_{m+1}(\alpha|\varpi)\}$$

* loc. cit., § 10, p. 449.

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= the coefficient of z^{m+2} in the expression of

$$\frac{z^2 e^{-\alpha z}}{1-e^{-\varpi z}}\left\{\frac{1}{1-e^{-2\pi\iota z}}+\frac{1}{1-e^{2\pi\iota z}}-1\right\}=0.$$

Therefore $\log L(\alpha)$ is, when $|\alpha|$ is very large and $\mathfrak{R}(\alpha)$ is positive, zero to an order greater than that of any power of $1/\alpha$. It is, of course, zero like $e^{-\alpha}$ and higher powers.

As a corollary note that we have incidentally shewn that

$$\Sigma_2 S'_{m+1}(\alpha\,|\,\varpi, \pm 2\pi\iota) = -{}_1S'_{m+1}(\alpha\,|\,\varpi).$$

§ 22. At this stage I conclude my investigations. We might, of course, apply the integral formulæ and transformation theory of Gamma-functions to Lambert's function. But the results do not seem to be of sufficient importance or elegance to deserve publication.

THE GENERALISATION OF THE MACLAURIN SUM FORMULA,* AND THE RANGE OF ITS APPLICABILITY.

By E. W. Barnes.

§ 1. THE fundamental asymptotic expansion which defines the Riemann ζ-function† is

$$\sum_{m=0}^{n-1}\frac{1}{(a+m\omega)^s}=\zeta(s,a)+\sum_{m=0}^{\infty}\frac{S_m'(a)}{m!}\left\{\frac{d^m}{dx^m}\frac{x^{1-s}}{1-s}\right\}_{x=n\omega}.$$

In this expansion n is large, a, ω, and s may have any complex values. The many valued functions which occur with s as index are such that $z^s=\exp.\{s\log z\}$, where the logarithm has its principal value with respect to the axis of $-\omega$.

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* In this paper I wish to correct and extend a form of the Maclaurin sum formula given in two previous papers:

(*a*) 'A Memoir on Integral Functions,' *Phil. Trans. Roy. Soc.* (A), Vol. CXCIX., pp. 411–500, § 41.

(*b*) 'The classification of Integral Functions,' *Camb. Phil. Trans.*, Vol. XIX, pp. 322–355, § 4. These papers will be referred to as I.F. and C.I.F. respectively.

† 'Theory of the Gamma Function,' *Messenger of Mathematics*, Vol. XXIX., pp. 64–128, §§ 23 and 35.

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When s is a negative integer equal to $-k$, say, we obtain

$$\sum_{m=0}^{n-1}(a+m\omega)^k=\zeta(-k,a)+\sum_{m=0}^{k+1}\frac{S_m'(a)}{m!}\left\{\frac{d^m}{dx^m}\frac{x^{k+1}}{k+1}\right\}_{x=n\omega}.$$

and we note that*

$$\zeta(-k,a)=-\frac{S'_{1+k}(a)}{1+k}.$$

Suppose now that $\phi(x)$ be an integral function of x of order less than unity. It will admit an expansion

$$\phi(x)=\sum_{k=0}^{\infty}a_kx^k,$$

and we shall have, when k is large, $|a_k|=\frac{1}{(k!)^{1-\epsilon}}$, where $\epsilon>0$.

Hence we have formally the asymptotic expansion

$$\sum_{m=0}^{n-1}\phi(a+m\omega)=-\sum_{k=0}^{\infty}a_k\frac{S'_{1+k}(a)}{1+k}+\sum_{m=0}^{\infty}\frac{S_m'(a)}{m!}\left[\frac{d^m}{dx^m}\int_0^x\phi(x)\right]_{x=n\omega}\quad\ldots(A).$$

First, the series $\sum_{k=0}^{\infty}a_k\frac{S'_{1+k}(a)}{1+k}$ is absolutely convergent. For $\frac{S_k'(a)}{k!}$ is the coefficient of $(-)^{k-1}z^k$ in the expansion of $\frac{(-z)e^{-az}}{1-e^{-\omega z}}$ in ascending powers of z. This expansion has a radius of convergence equal to $\frac{2\pi}{|\omega|}$, and therefore, when k is large,

$$\underset{k=0}{\text{Lt.}}\left[\frac{S_k'(a)}{k!}\right]^{\frac{1}{k}}=\frac{|\omega|}{2\pi},$$

so that $\sum_{k=0}^{\infty}a_k\frac{S'_{1+k}(a)}{1+k}$ is convergent with $\sum_{k=0}^{\infty}|a_k|k!\frac{|\omega|^{k+1}}{(2\pi)^{k+1}}$, and this is absolutely convergent if $|a_kk!|=\frac{1}{(k!)^{\epsilon}}$, $\epsilon>0$.

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* *Ibid*, § 27.

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Again in obtaining the formula (A) we have changed the order of summation in the series

$$\sum_{k=0}^{\infty} \sum_{m=0}^{k+1} a_k \frac{S_m'(a)}{m!} \frac{k!}{(k+1-m)!} x^{k+1-m},$$

and we have written it in the form

$$\sum_{m=0}^{\infty} \frac{S_m'(a)}{m!} \sum_{k=m-1}^{\infty} a_k \frac{k!}{(k+1-m)!} (n\omega)^{k+1-m}$$

$$= \sum_{m=0}^{\infty} \frac{S_m'(a)}{m!} \sum_{l=0}^{\infty} a_{l+m-1} \frac{(l+m-1)!}{l!} (n\omega)^l.$$

This process is legitimate* if the series

$$\sum_{l=0}^{\infty} a_{l+m-1} \frac{(l+m-1)!}{l!} (n\omega)^l$$

are convergent for all values of m, and if the double series last written is absolutely convergent.

Now, since, when k is large, $|a_k| = 1/\{k!\}^{1+\epsilon}$, where $\epsilon > 0$, we see that

$$\sum_{l=0}^{\infty} a_{l+m-1} \frac{(l+m-1)!}{l!} (n\omega)^l$$

is absolutely convergent with $\sum_{l=0}^{\infty} \frac{|(n\omega)^l|}{l!\{(l+m-1)!\}^{\epsilon}}$. The latter series is absolutely convergent, and when multiplied by $\{(m-1)!\}^{\epsilon-\eta}$, where $\epsilon > \eta > 0$ tends to zero as m tends to infinity.

Hence the double series is absolutely convergent with

$$\Sigma \frac{S_m'(a)}{m!} \frac{1}{\{(m-1)!\}^{\epsilon-\eta}},$$

and therefore with

$$\Sigma \left(\frac{|\omega|}{2\pi}\right)^m \frac{1}{\{(m-1)!\}^{\epsilon-\eta}}.$$

* Chrystal's *Algebra*, Vol. II., pp. 147–158.

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Hence the formula (A) constitutes an absolute and not an asymptotic equality when $\phi(x)$ is an integral function of order less than unity.

The formula (A) is the Maclaurin sum formula for a single parameter.

§ 2. We have only proved this formula under the limitation that $\phi(x)$ is an integral function of order less than unity.

Suppose, however, that $\phi(x)$ is an integral function of finite order greater than, or equal to, unity. The points $a+m\omega$ $[m=0, 1, \ldots, (n-1)]$ all lie within the region within which $\phi(x)$ can be expanded in the Taylor's series $\sum_{k=0}^{\infty} a_k x^k$; and evidently $\sum_{m=0}^{n-1} \phi(a+m\omega)$ will admit an asymptotic expansion when n is large.

The question arises as to whether the formula (A) fails to give this asymptotic expansion. *It does not fail to do so* provided we can regard the series $\sum_{k=0}^{\infty} a_k \frac{S'_{1+k}(a)}{1+k}$ as the summable divergent expansion of a definite function which is to be evaluated by the usual rules.*

For we have as before

$$\sum_{m=0}^{n-1} \phi(a+m\omega) = \sum_{k=0}^{\infty} a_k \left[\zeta(-k, a) + \sum_{m=0}^{k+1} \frac{S_m'(a)}{m!} \frac{k!}{(k+1-m)!} (n\omega)^{k+1-m} \right],$$

and this will be equal to

$$\sum_{k=0}^{\infty} a_k \zeta(-k, a) + \sum_{m=0}^{\infty} \frac{S_m'(a)}{m!} \left[\sum_{l=0}^{\infty} a_{l+m-1} \frac{(l+m-1)!}{l!} (n\omega)^l \right],$$

provided we can assign any meaning to the transformed series which shall make the process of transformation valid.

Now the series first written is equal to

$$\int_0^{\infty} e^{-x} \sum_{k=0}^{\infty} \frac{a_k x^k}{k!} \left[\zeta(-k, a) + \sum_{m=0}^{k+1} \frac{S_m'(a)}{m!} \frac{k!}{(k+1-m)!} (n\omega)^{k+1-m} \right] dx,$$

* The theory of summable divergent series was first given by Borel, *Liouville*, 5 Sér. t. 2, pp. 103 et seq.: *Ann. de l'Ecole Normale Superieure*, 3 Sér. t. 16, pp. 1 et seq It has been developed by the author, I.F. Part II.

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and this is equal to

$$\int_0^\infty e^{-x}\left\{\sum_{k=0}^{\infty}\frac{a_k}{k\,!}\zeta(-k,a)\,x^k+\sum_{m=0}^{\infty}\frac{S_m'(a)}{m\,!}\sum_{l=0}^{\infty}a_{l+m-1}\frac{(n\omega)^l}{l\,!}x^{l+m-1}\right\}dx.$$

It is legitimate to change the order of the double series under the sign of integration, since, if $\phi(x)$ is an integral function of finite order, $|a_k|=\frac{1}{\{k\,!\}^\epsilon}$, where $\epsilon>0$, when k is large, and

$$\sum_{l=0}^{\infty}a_{l+m-1}\frac{(n\omega)^l}{l\,!}x^{l+m-1}$$

is at most divergent with

$$\sum_{l=0}^{\infty}\frac{|xn\omega|^l}{l\,!}\frac{|x^{m-1}|}{\{(l+m-1)\,!\}^\epsilon},$$

and therefore with

$$\frac{|x^{m-1}|}{(m-1!)^\epsilon}\sum_{l=0}^{\infty}\frac{|xn\omega|^l}{l\,!},$$

and the last series is absolutely convergent.

Thus the double series last written under the sign of integration is absolutely convergent with

$$\sum_{m=1}^{\infty}\frac{|S_m'(a)|\,|x^{m-1}|}{m\,!\,\{(m-1)\,!\}^\epsilon}\sum_{l=0}^{\infty}\frac{|xn\omega|^l}{l\,!},$$

and is therefore absolutely convergent.

If now we define the two summable divergent series in the formula (A) as equal to the integrals by which they have been represented whenever the latter have a meaning the formula will hold for all cases in which $\phi(x)$ is an integral function of finite order.

It may be objected that the formula which needs such a cumbrous interpretation is useless. The examples which follow may shew that this is not necessarily the case. And it has been established [I.F., § 30] that such series 'sum' to the function of which they are the arithmetically asymptotic expansion according to Poincaré's definition.

The integrals employed or the contour integrals by which they can be replaced, to obtain a more general range of existence [I.F., § 24] will not have a meaning for particular

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values of a and ω [I.F., § 26], and for such values the interpretation of the formula fails. But we can shew that the two integrals when taken together have a meaning, and therefore that when and only when the interpretation fails to 'sum' $\sum_{k=0}^{\infty} a_k \zeta(-k, a)$ will it fail to interpret the remaining asymptotic expansion. For the second integral is equal to the first and this in turn is equal to

$$\sum_{m=0}^{n-1} \int_0^{\infty} \left\{ e^{-x} \sum_{k=0}^{\infty} \frac{a_k (a+m\omega)^k}{k!} x^k \right\} dx.$$

The series inside the bracket is always of order less than unity, and therefore the integral (or the contour integral by which it may be replaced) has always a meaning.

§ 3. The extension of the Maclaurin sum formula to the case when $\phi(x)$ is not of order less than unity suggested in the author's previous papers is invalid for the following reason.

It is true that, when $\phi(x)$ is such that the series $\sum_{s=0}^{\infty} \frac{a_s s!}{(-z)^{s+1}}$ is convergent and represents the Taylor's expansion of a function $\chi(z)$ outside a circle of finite radius ρ, we can take

$$\sum_{m=0}^{n-1} \phi(a+m\omega) = \frac{\iota}{2\pi} \int \frac{e^{-az} - e^{-(a+n\omega)z}}{1-e^{-\omega z}} \chi(z)\, dz,$$

when the contour of the integral is such that its bulb encloses the circle whose centre is the origin and radius ρ. Yet we cannot expand $\frac{ze^{-(a+n\omega)z}}{1-e^{-\omega z}}$ under the sign of contour integration in a series of ascending powers of z when the contour encloses poles of this expression. For the expansion is divergent outside a circle of radius $2\pi/|\omega|$, and this divergent expansion is only summable to $\frac{ze^{-(a+n\omega)z}}{1-e^{-\omega z}}$ within the simply connected region bounded by lines proceeding from its poles in a direction away from the origin to infinity. And when the bulb of the contour encloses the circle of radius ρ and ρ is greater than $2\pi/|\omega|$, the contour is no longer within this simply connected region, and the fundamental procedure is invalid.

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§ 4. The Maclaurin sum formula (A) is valid in an even more extended range of cases than when $\phi(x)$ is an integral function of any order. *It is valid whenever the Taylor's series for* $\phi(x)$, *viz.* $\sum_{k=0}^{\infty} a_k x^k$, *gives the summable though divergent expansion of* $\phi(x)$ *for* $x = a + m\omega$ $(m = 0, 1, \ldots, (n-1))$. For in such cases a suitable modification of the previous argument continues to apply. Thus it is valid [I.F., § 26] whenever $\phi(x)$ is a possibly non-uniform function with a finite number of poles or logarithmic singularities in the finite part of the plane. The search for further ranges of validity of the formula is reduced by the theorem just stated to the investigation of the cases in which a Taylor's series outside its circle of convergence is the summable divergent expansion of the function which the series represents when convergent.

We proceed to give two concrete examples of the theorems just stated. In the first example $\phi(x)$ is of order unity; in the second it is a non-uniform function represented by a series convergent within a circle of radius $|\omega|$.

§ 5. The first case for which the series $\sum_{k=0}^{\infty} a_k \frac{S'_{1+k}(a)}{1+k}$ is not convergent occurs when $\phi(x)$ is of order unity.

Let us consider then what happens when $\phi(x) = e^{lx}$.

The formula (A) becomes

$$e^{la} + e^{l(a+\omega)} + \ldots + e^{l\{a+(n-1)\omega\}} = -\sum_{k=0}^{\infty} \frac{l^k}{k!} \frac{S'_{1+k}(a)}{1+k} + \sum_{m=0}^{\infty} \frac{S_m'(a)}{m!} \left[\frac{d^m}{dx^m} \frac{e^{lx}-1}{l}\right]_{x=n\omega}.$$

Now the series

$$-\sum_{k=0}^{\infty} \frac{l^k}{k!} \frac{S'_{1+k}(a)}{1+k}$$

is convergent if $|\omega l| < 2\pi$, and is then equal to

$$\frac{e^{al}}{1-e^{\omega l}} + \frac{S_0'(a)}{l}.$$

If $|\omega l| \not< 2\pi$, the series is divergent, but regarded as a

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summable divergent series it is still equal to the same expression except for values of l lying on the lines drawn from the points $\pm\frac{2\pi\iota}{\omega}$ away from the origin to infinity. It is infinite if $\omega l = 2\nu\pi\iota$ (ν being a non-zero integer).

Let us examine the second asymptotic series

$$\sum_{m=0}^{\infty} \frac{S_m'(a)}{m!} \left[\frac{d^m}{dx^m} \frac{e^{lx}-1}{l}\right]_{x=\omega n}.$$

It is equal to

$$-\frac{S_0'(a)}{l} + e^{n\omega l} \sum_{m=0}^{\infty} \frac{S_m'(a)}{m!} l^{m-1}.$$

The series just written is divergent if $|\omega l| \not< 2\pi$, but it will 'sum' (under similar restrictions) to

$$-\frac{e^{al}}{1-e^{\omega l}}.$$

Now $$e^{la} + \ldots + e^{l\{a+(n-1)\omega\}} = e^{la}\frac{1-e^{ln\omega}}{1-e^{l\omega}}.$$

Hence the Maclaurin sum formula applied to the case $\phi(x) = e^{lx}$ gives us for all values of $|l|$, the obvious equality

$$e^{la}\frac{1-e^{ln\omega}}{1-e^{l\omega}} = \frac{e^{la}}{1-e^{l\omega}} - \frac{e^{la+ln\omega}}{1-e^{l\omega}}.$$

We may therefore say that*

$$\sum_{m=0}^{n-1} e^{l(a+m\omega)} = \frac{e^{al}}{1-e^{\omega l}} + \frac{S_0'(a)}{l} + \sum_{m=0}^{\infty} \frac{S_m'(a)}{m!} \left[\frac{d^m}{dx^m} \frac{e^{lx}-1}{l}\right]_{x=\omega n},$$

for all values of l. And even when the first term becomes infinite there will be a countervailing infinity which will arise in summing the asymptotic series and render the formula, though useless, still valid.

§ 6. As a further example we will consider a case when $\phi(x)$ has a finite radius of convergence.

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* The formula has been previously employed to obtain the asymptotic expansion of the standard integral function of zero order [*Vide* I.F., § 61 and C.I.F., § 12].

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We take

$$\phi(x) = \log(\omega + x) = \log\omega + \frac{x}{\omega} - \frac{x^2}{2\omega^2} + \ldots.$$

The series is divergent but summable in the dissected region outside a circle of radius $|\omega|$. [I.F., § 26]. The formula (A) when applied to this case becomes

$$\sum_{m=0}^{n-1} \log(a + \omega + m\omega)$$

$$= Z + \sum_{m=0}^{\infty} \frac{S_m'(a)}{m!} \frac{d^m}{dx^m} [(x+\omega)\log(x+\omega) - x - \omega\log\omega]_{x=n\omega},$$

where $$Z = -\log\omega\, S_1'(a) + \sum_{s=1}^{\infty} \frac{(-)^s S'_{s+1}(a)}{s(s+1)\omega^s}.$$

This formula may be written

$$\sum_{m=0}^{n-1} \log(a + m\omega) = Z + \log a + S_0'(a)\{\omega - \omega\log\omega\}$$

$$+ \sum_{m=0}^{\infty} \frac{S_m'(a)}{m!} \frac{d^m}{dx^m} [x\log x - x]_{x=n\omega}.$$

In order to give it a practical meaning, we must sum the divergent series

$$\sum_{s=1}^{\infty} \frac{(-)^s S'_{s+1}(a)}{s(s+1)\omega^s},$$

which enters into the expression for Z.

In order to avoid analytical difficulties which do not concern the point with which we are dealing we will assume that $\mathfrak{R}(\omega)$ and $\mathfrak{R}\left(\frac{a}{\omega}\right)$ are both positive. The series is an example of the case in which we must duplicate Borel's process of summation, as extended by myself [I.F., §§ 32 and 33]. Its value is

$$\int_0^{\infty} e^{-x} \frac{\omega}{x^2} \sum_{s=1}^{\infty} \frac{(-)^s S'_{s+1}(a)}{(s+1)!} \left(\frac{x}{\omega}\right)^{s+1} dx,$$

whenever the integral, which may be taken along any line in the positive half of the plane, has any meaning.

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Now the series

$$\sum_{s=1}^{\infty} \frac{(-)^s S'_{s+1}(a)}{(s+1)!} \left(\frac{x}{\omega}\right)^{s+1}$$

has no meaning when $\left|\frac{x}{\omega}\right| > \left|\frac{2\pi}{\omega}\right|$ unless we employ a similar process to sum it. Employing such a process, its value is

$$\frac{-\frac{x}{\omega} e^{-a\frac{x}{\omega}}}{1-e^{-x}} + S_0'(a) - S_1'(a)\frac{x}{\omega}.$$

The integral is therefore equal to

$$\int_0^{\infty} \left\{ \frac{-e^{-x\left(1+\frac{a}{\omega}\right)}}{x(1-e^{-x})} + S_0'(a)\frac{\omega e^{-x}}{x^2} - \frac{S_1'(a)}{x} e^{-x} \right\} dx$$

$$= \int_0^{\infty} \left\{ \frac{-e^{-x(a+\omega)}}{x(1-e^{-\omega x})} + S_0'(a)\frac{e^{-\omega x}}{x^2} - \frac{S_1'(a)}{x} e^{-\omega x} \right\} dx.$$

If the former integral was taken along the positive direction of the real axis, the latter must be taken along the positive direction of the axis of $1/\omega$.

Its value is therefore equal to*

$$-\log\frac{\Gamma_1(a+\omega)}{\rho_1(\omega)} - \int_0^{\infty} \left(\frac{1}{\omega}\right) \frac{S_1'(a)}{x} \{e^{-\omega x} - e^{-x}\} dx$$

$$+ \int_0^{\infty} \left(\frac{1}{\omega}\right) \left\{ \frac{e^{-\omega x} - 1}{\omega x^2} + \frac{e^{-x}}{x} \right\} dx.$$

since $\qquad S_0'(a) = \frac{1}{\omega}, \quad S_1'(a+\omega) - S_1'(a) = 1.$

Now† if $\mathfrak{R}(\omega)$ is positive

$$\int_0^{\infty} \frac{e^{-\omega x} - e^{-x}}{x} dx = -\log\omega.$$

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* "Theory of the Gamma-function," § 31. [There is an obvious misprint in the formula quoted]

† *Ibid*, § 31.

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Also if ε be a point on the axis of $1/\omega$ very near the origin

$$\int_\varepsilon^\infty \frac{e^{-\omega x}-1}{\omega x^2}\,dx = \log\varepsilon + \log\omega - 1 + \gamma$$

$$+ \text{terms which vanish with } |\varepsilon|,$$

$$\int_\varepsilon^\infty \frac{e^{-x}}{x}\,dx = -\log\varepsilon - \gamma + \text{similar terms.}^*$$

Hence $$\int_0^\infty \left(\frac{1}{\omega}\right)\left\{\frac{e^{-\omega x}-1}{\omega x^2}+\frac{e^{-x}}{x}\right\}dx = \log\omega - 1.$$

Hence $$Z + \log a + S_0'(a)\{\omega - \omega\log\omega\}$$

$$= -\log\frac{\Gamma_1(a+\omega)}{\rho_1(\omega)} + \log a = -\log\frac{\Gamma_1(a)}{\rho_1(\omega)}.$$

We therefore have the asymptotic equality

$$\sum_{m=0}^{n-1}\log(a+m\omega) = -\log\frac{\Gamma_1(a)}{\rho_1(\omega)} + \sum_{m=0}^{\infty}\frac{S_m'(a)}{m!}\frac{d^m}{dx^m}[x\log x - x]_{x=\omega},$$

which was previously obtained by an entirely different procedure.†

The two examples which have been given are perhaps sufficient to show the application of the previous extensions of the applicability of the Maclaurin sum-formula.

§ 7. We may now indicate a further extension.

Suppose that $\phi(x)$ has an essential singularity at the origin, but no other singularity at any other point of the plane (infinity included). Then outside a point circle surrounding the origin it admits the Laurent expansion

$$\phi(x) = a_0 + \frac{a_1}{x} + \ldots + \frac{a_k}{x^k} + \ldots,$$

and we have approximately, when k is large, $|a_k| = \frac{1}{(k!)^{\varepsilon}}$ where $\varepsilon > 0$.

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* *Ibid*, § 28 Cor. † *Ibid*, § 30.

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Now* if s be a positive integer > 1

$$\sum_{m=0}^{n-1}\frac{1}{(a+m\omega)^s}=\frac{(-)^s}{(s-1)!}\frac{d^s}{da^s}\log\Gamma_1(a)+\sum_{m=0}^{\infty}\frac{S_m'(a)}{m!}\frac{d^m}{dx^m}\left(\frac{x^{1-s}}{1-s}\right)_{x=n\omega},$$

and

$$\sum_{m=0}^{n-1}\frac{1}{a+m\omega}=-\frac{d}{da}\log\Gamma_1(a)+\sum_{m=0}^{\infty}\frac{S_m'(a)}{m!}\frac{d^m}{dx^m}(\log x)_{x=n\omega}.$$

Let now

$$\int^x\phi(x)\,dx=a_0x+a_1\log x-\frac{a_2}{x}-\frac{a_3}{2x^2}-\ldots,$$

so that $\phi(x)$ is to be expanded in a semi-Laurent series under the sign of integration, and the lower limit of the integral is 0, 1 or ∞ according as the corresponding term where integrated vanishes for such a limit.

Then we have $(a\neq 0)$

$$\sum_{m=0}^{n-1}\phi(a+m\omega)=-a_0S_1'(a)+\sum_{k=1}^{\infty}\frac{(-)^ka_k}{(k-1)!}\frac{d^k}{da^k}\log\Gamma_1(a)$$
$$+\sum_{m=0}^{\infty}\frac{S_m'(a)}{m!}\left[\frac{d^m}{dx^m}\int^x\phi(x)\,dx\right]_{x=n\omega}.$$

The identity is evidently formally true. Also within the least circle of radius $|a+r\omega|$, where r is a positive integer, we have the expansion

$$\log\Gamma_1(a+z)=\log\Gamma_1(a)+\sum_{k=1}^{\infty}\frac{z^k}{k!}\frac{d^k}{da^k}\log\Gamma_1(a).$$

Hence, approximately, when k is large,

$$\frac{1}{k!}\frac{d^k}{da^k}\log\Gamma_1(a)=\frac{1}{|(a+r\omega)^k|}.$$

Therefore $\sum_{k=1}^{\infty}\frac{(-)^ka_k}{(k-1)!}\frac{d^k}{da^k}\log\Gamma_1(a)$ is convergent when when $|a_k|=1/(k!)^s$ and $s>0$. The first series on the right-hand side of the quality is therefore convergent, and the

* 'The Theory of the Gamma-Function,' § 27.

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theorem may be completed by giving integral expressions for the asymptotic series employed. The essential difference between this case and that of § 2 is that the fundamental series are now asymptotic instead of being finite, and therefore we must initially employ a contour integral for their interpretation.

Thus, if we ignore the infinity which arises when $\theta = 0$,

$$\sum_{m=0}^{n-1} \phi(a+m\omega)$$

$$= \operatorname*{Lt}_{\theta=0} \int_0^\infty e^{-y} \sum_{s=0}^{\infty} a_s \sum_{m=0}^{\infty} (-)^m \frac{S_m'(a)}{m!} x^{1-s-m-\theta} \frac{y^{\frac{1}{2}(s+m+\theta)-1} \sqrt{\pi}\, 2^{1-\frac{1}{2}(s+m+\theta)}}{\Gamma(s+m+\theta+\frac{1}{2})} dy,$$

and we may interchange the order of summation in the double series, since it is absolutely convergent. The other processes previously suggested [I.F., § 37] will also serve to interpret the asymptotic series.

And moreover, if $\phi(x)$ admit a semi-Laurent expansion outside a circle of finite radius inside which this expansion is a summable divergent series for the function, and if the points $a+m\omega$ do not lie on the lines joining the origin to the singularities of the function, the preceding formula is still valid provided the series $\sum_{k=1}^{\infty} \frac{(-)^k a_k}{(k-1)!} \frac{d^k}{da^k} \log \Gamma, (a)$ is regarded as a divergent series summable by the usual rules.

§ 8. Combine now the theorems which have been stated and we see that if $\phi(x)$ admits a Laurent expansion within a ring space surrounding the origin, and if this expansion exists as a summable divergent series within the simply connected area formed by joining singularities inside the inner circle of the ring space to the origin, and outside the outer circle to infinity (the latter lines being in a direction away from the origin), and if the points $a+m\omega$ are all situated within this area, we have the formula

$$\sum_{m=0}^{n-1} \phi(a+m\omega) = \zeta(\phi) + \sum_{m=0}^{\infty} \frac{S_m'(a)}{m!} \left[\frac{d^m}{da^m} \int^x \phi(x)\, dx\right]_{x=n\omega},$$

where the function $\zeta(\phi)$ is given, in general, by a summable divergent series.

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§ 9. The same remarks which have been made in the case of the Maclaurin sum formula apply to my extension to the case of two parameters [C.I.F., § 4].

It is possible, as I have shown in a paper shortly to be published, to extend the analysis to the case of any number of parameters. We have in fact

$$\sum_{m_1=0}^{n_1-1} \dots \sum_{m_r=0}^{n_r-1} \phi(a+m_1\omega_1+\dots+m_r\omega_r)$$

$$= \sum_{s=0}^{\infty} \frac{(-)^r a_s {}_rS'_{s+1}(a)}{s+1} + \sum_{m=0}^{\infty} \frac{{}_rS_m^{(r)}(a)}{m!} F_r\left[\frac{d^m}{dx^m}\psi(x)\right]_{x=\infty},$$

where $\phi(x)$ admits the expansion $\sum_{s=0}^{\infty} a_s x^s$; the n's are large; the functions ${}_rS_m(a)$ are r-ple Bernoullian functions;

$$\psi(x) = \int_0^x dx \int_0^x dx \dots \int_0^x \phi(x)\, dx$$

(the integration being r times repeated), and where $F_r[f(x)]_{x=\infty}$ is a symbolic way of writing

$$f(n_1\omega_1+\dots+n_r\omega_r) - \sum_{*=1}^{r} f(n_1\omega_1+\dots+*+\dots+n_r\omega_r)$$

$$+ \sum_{*=1}^{r}\sum_{*=1}^{r} f(n_1\omega_1+\dots+*+\dots+*+\dots+n_r\omega_r) - \dots + (-)^{r-1} \sum_{k=1}^{r} f(n_k\omega_k).$$

In the first summation the star denotes that one of the $n\omega$'s is to be omitted: in the second summation every two different pairs of $n\omega$'s are to be successively omitted, and so on.

When $\phi(x)$ is of order less than unity, the series

$$\sum_{s=0}^{\infty} \frac{(-)^r a_s {}_rS'_{s+1}(a)}{s+1}$$

is convergent. In other cases it is to be regarded as a summable divergent expansion which may be evaluated by the usual rules.

An obvious modification gives a formula which is applicable to the general case when $\phi(x)$ is given by a Laurent's series, and the conditions of the previous paragraph hold.

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A NOTE ON THE CONDITION OF INTEGRABILITY OF A FUNCTION OF A REAL VARIABLE.

By W. H. YOUNG.

§ 1. THE usual mode of stating the condition of integrability of a function of a real variable is the following:—The necessary and sufficient condition for the integrability of the finite function $f(x)$ in the interval (A, B) is that, given any positive quantity k, the points at which the degree of discontinuity is $\geq k$ form a closed set of zero content.

Stated in this form the integrability of the function appears as if it were dependent on the actual values of the function in the interval for which it is defined. The object of the present note is to point out that this is not the case, and to modify the form of the condition accordingly. Incidentally I have been led to a new theorem in the theory of Sets of Points, which, for the purposes in hand, is only required in a simplified form, and is given in full at the end of the paper.

§ 2. Lemma:—

If $G_1, G_2, \ldots$ *be closed sets, of zero content, such that each is contained in all the succeeding sets, the limiting set* G,* (*in general open*), *possesses the property that every closed component of* G *has zero content.*

For, assigning any small positive quantity e, we can describe round every point of any set G_n as centre an interval, so that the content† of the whole set of intervals is less than $\frac{e}{2^{n+1}}$.

Now, P being any point of G, we can, (by the definition of G), assign a definite integer i, such that, for all values of $n \geq i$, and for no value of $n < i$, P is a point of G_n. Round each such point P, therefore, we take that one of the intervals specified above which corresponds to P, considered as a point of G_i. The content of all such intervals, corresponding to every point of G, (being certainly less than the sum of the contents of all such sets of intervals, which is $e(\frac{1}{2}+\frac{1}{4}+\ldots)$, or e), is certainly less than e. Thus the points of any closed set contained in G are shut up in a set of intervals of content less than e, so that, by a known property often taken as the

* G is defined by the two properties: (1) every point of G_n is a point of G for all values of n; (2) no point of G exists which does not belong to some definite G_n of the sequence.

† That is, the sum of the lengths of the equivalent set of non-overlapping intervals. See W. H. Y., *Proc. L. M. S.* 35, p 386.

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definition of the content, the content of such a closed component of G is not greater than e. But e could be chosen as small as we please, so that, the content of such a component, being smaller than any assignable quantity, cannot be other than zero. Q.E.D.

§ 3. *Take now any set of points having the following property:—the complementary set is the limiting set of a sequence of closed sets of zero content; then we assert that any finite function having these points for its points of continuity* is integrable, and secondly that every finite integrable function is of this type.*

For, by the Lemma, the content of every closed component of this complementary set will be zero; and it is known that the points at which the degree of discontinuity is $\geq k$ form a closed component of this set, hence the conditions required as as stated at the commencement of this note are satisfied, and therefore the function is integrable, That every finite integrable function is of this form is an immediate consequence of the condition in its old form.

We may remark further, that if the function whose integrability is in question has other points of continuity beside those common to both functions, the function is still integrable.

Cor. Any two finite functions having the same points of continuity being given, if one be integrable, so is the other.

With our formulation of the condition of integrability, Du Bois Reymond's important theorem—*that a continuous function of a finite number of integrable functions is itself integrable*—of which the only proof with which I am acquainted was given by Du Bois Reymond himself, *Math. Ann.* 20, p. 123,—is almost intuitive.

In fact, let $f(u_1, u_2, \ldots, u_n)$ be the function in question, $u_1, u_2, \ldots, u_n$ being integrable functions of x, and f being a continuous function of the u's. Then the points of the interval in question for which the u's are all continuous, are points of continuity of the function. Hence the only possible points of discontinuity are those of the functions $u_1, u_2, \ldots, u_n$. Assume that they are all points of discontinuity of f. Then the points of discontinuity consist of a finite number of limiting sets, of the kind described in the enunciation of the theorem of the preceding paragraph.

* That such functions always exist has been proved by me elsewhere.

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But the sum of a finite number of closed sets of zero content is itself a closed set of zero content. Hence the points of discontinuity of f form a set of the right type, and the theorem follows. If the points are not all points of discontinuity of f, then the result still follows, by the remark made at the conclusion of the preceding paragraph.

§ 5. The lemma stated and proved in § 2 is a special case of the following theorem, which may be regarded as a further contribution to the at present incomplete theory of the content of open sets:—

*If $G_1, G_2, \ldots$ be a countably infinite sequence of closed sets, dense nowhere, such that each is contained in all the succeeding sets, and I the limit of the content of G_n when n is infinite, the content of any closed component of the limiting set G cannot exceed I.**

Let the content of G_n be denoted by I_n, and the length of the whole segment by l; then $(l - I_n)$ is the content of the black intervals† of G_n. Also

$$I \geq I_n \geq I_{n-1}.$$

Assign any small quantity e.

Take a finite number of the black intervals of G_1, such that their content lies between $(l - I_1)$ and $\left(l - I_1 - \frac{e}{4}\right)$, and cut off from each end of each of these a fraction $\frac{e}{2^3(l - I_1)}$ of its length. The content of these parts being less than $\frac{e}{4}$, and the same being true of the remaining black intervals of G_1, the content of the curtailed intervals lies between $\left(l - I_1 - \frac{e}{2}\right)$ and $(l - I_1)$, *a fortiori* their content is greater than $(l - I - e)$, and they contain no point of G_1 as internal or end-point. Outside these we have a finite number of non-overlapping intervals, containing all the points of G_1, whose content lies between $\left(I_1 + \frac{e}{2}\right)$ and I_1.

Now, since the black intervals of G_2 lie inside those of G_1, inside the above curtailed intervals there will be black intervals

* Cp. Osgood's theorem that when G is closed its content is I, *Amer. Journ.*, 19, also W. H. Y., *Proc. L. M. S.*, 35, 269.

† That is the non-overlapping intervals free of points of G_n, whose end points are points of G_n.

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and parts of black intervals of G_2, whose content lies between $\left(l-I_2-\frac{e}{2}\right)$ and $(l-I_2)$. Among these we can choose out a finite number whose content lies between $\left(l-I_2-\frac{e}{2}-\frac{e}{8}\right)$ and $(l-I_2)$, and curtail each at each end by a fraction $\frac{e}{2^4(l-I_2)}$ of its length; the content of these curtailed intervals lies between $\left(l-I_2-\frac{e}{2}-\frac{e}{4}\right)$ and $(l-I_2)$; *a fortiori* the content is greater than $(l-I-e)$, and they contain no point of G_2 as internal or end point. Outside them we have a finite number of non-overlapping intervals (consisting of the former ones, possibly lengthened, and possibly new intervals), containing all the points of G_2, and of content lying between $\left(I_2+\frac{e}{2}+\frac{e}{4}\right)$ and I_2.

Proceeding thus in turn with G_3, G_4, ... we construct at each stage such a finite number of intervals, and since each interval so constructed is either lengthened or left unaltered at the next stage, so that it has a definite limiting interval, we determine in this way a definite set of non-overlapping intervals whose content is not greater than the upper limit of

$$\left(I_n+\frac{e}{2}+\frac{e}{4}+\frac{e}{8}+\ldots+\frac{e}{2^n}\right)$$

for all values of n, and is therefore not greater than $(I+e)$, and containing every point of G as internal point. Now, by a known theorem* there are a finite number of these intervals containing the points of any closed component of G, and if J be the content of this closed component, the content of those intervals cannot be less than J. Hence J is not greater than $(I+e)$. Since e may be as small as we please this proves the theorem.

It may be noted that this theorem indicates the existence of unclosed sets, such that, though every closed set containing them has content as *large* as we please, every closed set contained in them has content as *small* as we please. In another place, (*loc. cit.*) I have given an example of such a set in the segment (0, 1), where the content of the set got by closing it lies between $\frac{1}{2}$ and $\frac{2}{3}$, while, by the theorem just proved, it contains only closed sets of content zero.

* See W. H. Y., *Proc. L. M. S.*, 35, p. 387, "Given any closed set of points on a straight line and a set of intervals so that every point of the closed set of points is an internal point of at least one of the intervals, then there exist a finite number of the given intervals having the same property."

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NOTE ON THE FUNCTION $\int_x^\infty e^{\frac{1}{2}(x^2-t^2)}\,dt$.

By G. H. Hardy, Trinity College, Cambridge.

§ 1. It is well-known* that

$$\int_0^\infty \frac{\sinh \mu x}{\sinh \frac{1}{2}x} \cos \alpha x\, dx = \frac{\pi \sin 2\mu\pi}{\cosh 2\alpha\pi + \cos 2\mu\pi},$$

$$\int_0^\infty \frac{\cosh \mu x}{\cosh \frac{1}{2}x} \cos \alpha x\, dx = \frac{2\pi \cosh \alpha\pi \cos \mu\pi}{\cosh 2\alpha\pi + \cos 2\mu\pi},$$

if α is real, and $0 \leqq \mu < \frac{1}{2}$.

If we multiply both sides of each of these equations by $e^{-\beta\alpha^2}$ and integrate from $\alpha = 0$ to $\alpha = \infty$, with the help of the formula

$$\int_0^\infty e^{-\beta\alpha^2} \cos \alpha x\, d\alpha = \tfrac{1}{2}\sqrt{\left(\frac{\pi}{\beta}\right)}\, e^{-\frac{x^2}{4\beta}},$$

we obtain

$$(1)\quad \int_0^\infty \frac{\sinh \mu x}{\sinh \frac{1}{2}x} e^{-\frac{x^2}{4\beta}}\, dx = 2\sqrt{(\beta\pi)}\, \sin 2\mu\pi \int_0^\infty \frac{e^{-\beta\alpha^2}\, d\alpha}{\cosh 2\alpha\pi + \cos 2\mu\pi},$$

and

$$(2)\quad \int_0 \frac{\cosh \mu x}{\cosh \frac{1}{2}x} e^{-\frac{x^2}{4\beta}}\, dx = 4\sqrt{(\beta\pi)}\, \cos \mu\pi \int_0^\infty \frac{e^{-\beta\alpha^2} \cosh \alpha\pi\, d\alpha}{\cosh 2\alpha\pi + \cos 2\mu\pi}.$$

In certain particular cases these formulæ take a symmetrical and very elegant form. If *e.g.* we make $\mu = \frac{1}{4}$ in (1), and put

$$x = 4\pi\xi, \quad \alpha = \tfrac{1}{2}\xi,$$

$$\frac{4\pi^2}{\beta} = a, \quad \tfrac{1}{4}\beta = b,$$

* See *e.g.* Harkness and Morley's *Introduction to the Theory of Analytic Functions*, pp. 226–229.

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we obtain

$$(3)\quad a^{\frac{1}{2}}\int_0^\infty \frac{e^{-a\xi^2}d\xi}{\cosh \pi\xi} = b^{\frac{1}{2}}\int_0^\infty \frac{e^{-b\xi^2}d\xi}{\cosh \pi\xi},$$

if $ab=\pi^2$. Similarly, by making $\mu=\frac{1}{6}$, so that

$$\frac{\sinh \mu x}{\sinh \frac{1}{2}x} = \frac{\sinh \frac{1}{6}x}{\sinh \frac{1}{2}x} = \frac{1}{1+2\cosh \frac{1}{3}x},$$

and putting

$$x=3\pi\xi, \quad \alpha=\tfrac{1}{2}\xi,$$

$$\frac{9\pi^2}{4\beta}=a, \quad \tfrac{1}{4}\beta=b,$$

we obtain

$$(4)\quad a^{\frac{1}{2}}\int_0^\infty \frac{e^{-a\xi^2}d\xi}{1+2\cosh \pi\xi} = b^{\frac{1}{2}}\int_0^\infty \frac{e^{-b\xi^2}d\xi}{1+2\cosh \pi\xi},$$

if $ab=(\frac{3}{4}\pi)^2$. If we make $\mu=\frac{1}{4}$ in (2), and put

$$x=2\pi\xi, \quad \alpha=\tfrac{1}{2}\xi,$$

$$\frac{\pi^2}{\beta}=a, \quad \tfrac{1}{4}\beta=b,$$

we obtain

$$(5)\quad a^{\frac{1}{2}}\int_0^\infty \frac{\cosh \frac{1}{2}\pi\xi}{\cosh \pi\xi} e^{-a\xi^2}d\xi = b^{\frac{1}{2}}\int_0^\infty \frac{\cosh \frac{1}{2}\pi\xi}{\cosh \pi\xi} e^{-b\xi^2}d\xi,$$

if $ab=(\frac{1}{2}\pi)^2$.

§ 2. These formulæ at once suggest certain formulæ proved by Cauchy, of which

$$\sqrt{a}\{\tfrac{1}{2}+e^{-a^2}+e^{-4a^2}+e^{-9a^2}+\ldots\} = \sqrt{b}\{\tfrac{1}{2}+e^{-b^2}+e^{-4b^2}+e^{-9b^2}+\ldots\}$$

$(ab=\pi)$ is typical. These are, of course, special cases of the formulæ for the linear transformation of the Theta-functions.

We may transform (3), (4), (5) in such a way as to render their analogy with Cauchy's formulæ more apparent. For it is easy to prove that, if

$$I_n(a) = \int_0^\infty \frac{e^{-a\xi^2}\cos 2na\xi}{\cosh \pi\xi}\,d\xi,$$

$$I_0(a) - (-)^n e^{n^2a} I_n(a) = \sum_{k=0}^{n-1} (-)^k e^{(k+\frac{1}{2})^2a};$$

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so that (3) takes the form

$$a^{\frac{1}{4}}\{e^{\frac{1}{4}a} - e^{\frac{3}{4}a} + e^{\frac{5}{4}a} - \ldots + (-)^{n-1} e^{(n-\frac{1}{2})^2 a} + (-)^n e^{n^2 a} I_n(a)\},$$

$$= b^{\frac{1}{4}}\{e^{\frac{1}{4}b} - e^{\frac{3}{4}b} + e^{\frac{5}{4}b} - \ldots + (-)^{n-1} e^{(n-\frac{1}{2})^2 b} + (-)^n e^{n^2 b} I_n(b)\}.$$

A similar transformation may be applied to (4) and (5). However, it seems that these formulæ cannot be themselves deduced from the theory of elliptic functions, but depend on the theory of the infinite products (such as the Double Gamma-function) out of which the elliptic functions may be built up. On the other hand I have been able to connect them with Cauchy's theory of *reciprocal functions* in a way which seems to me interesting.

§ 3. It follows from Fourier's double-integral theorem, that if

$$\psi(\alpha) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \cos \alpha x \, \phi(x) \, dx,$$

then

$$\phi(\alpha) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \cos \alpha x \, \psi(x) \, dx;$$

and if

$$\psi(\alpha) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0 \sin \alpha x \, \phi(x) \, dx,$$

then

$$\phi(\alpha) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \sin \alpha x \, \psi(x) \, dx,$$

provided $\phi(x)$ satisfies certain conditions. Cauchy calls $\phi(x)$ and $\psi(x)$ *reciprocal functions* of the first or of the second kind. Since

$$e^{-\frac{1}{2}\alpha^2} = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \cos \alpha x \, e^{-\frac{1}{2}x^2} \, dx,$$

the function $e^{-\frac{1}{2}x^2}$ is its own reciprocal of the first kind. So far as I am aware, Cauchy gives no example of a function which is its own reciprocal of the *second* kind.

This property is, as I shall show, possessed by the function

$$\phi(x) = \int_x^\infty e^{\frac{1}{2}(x^2 - t^2)} \, dt.$$

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§ 4. Before proving this it is essential to investigate the behaviour of $\phi(x)$ for very large values of x.

Now

$$\frac{1}{\sqrt{2}}\phi(x\sqrt{2}) = e^{x^2}\int_x^\infty e^{-t^2}dt = -\frac{1}{2x}\left\{xe^{x^2}\int_x^\infty \frac{1}{t}\frac{d}{dt}(e^{-t^2})\,dt\right\}$$

$$= \frac{1}{2x} - \tfrac{1}{2}e^{x^2}\int_x^\infty \frac{e^{-t^2}}{t^2}\,dt.$$

And as

$$e^{x^2}\int_x^\infty \frac{e^{-t^2}}{t^2}\,dt = -\tfrac{1}{2}e^{x^2}\int_x^\infty \frac{1}{t^3}\frac{d}{dt}(e^{-t^2})\,dt = \frac{1}{2x^3} - \tfrac{3}{2}e^{x^2}\int_x^\infty \frac{e^{-t^2}}{t^4}\,dt$$

and

$$e^{x^2}\int_0^\infty \frac{e^{-t^2}}{t^4}\,dt < \int_x^\infty \frac{dt}{t^4} = \frac{1}{3x^3}$$

it follows that

$$\frac{1}{\sqrt{2}}\phi(x\sqrt{2}) = \frac{1}{2x} + \frac{\rho}{x^3},$$

where ρ is a quantity whose modulus is less than $\frac{1}{2}$.

This process can evidently be repeated; in fact the asymtotic expansion of

$$e^{x^2}\int_x^\infty e^{-t^2}dt$$

is* $$\frac{1}{2x} - \frac{1}{2^2x^3} + \frac{1.3}{2^3x^5} - \frac{1.3.5}{2^4x^7} + \ldots.$$

It follows that

$$\int_0^\infty \sin\alpha x\,\phi(x)\,dx$$

is convergent. I shall now prove that the integral is equal to $\sqrt{\left(\frac{\pi}{2}\right)}\phi(\alpha)$.

---

* E. T. Whittaker, *Modern Analysis*, p. 166.

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§ 5. If

$$\psi(\alpha) = \int_0^\infty \sin \alpha x \, \phi(x)\, dx,$$

$$e^{-\frac{1}{2}\alpha^2} \psi(\alpha) = \int_0^\infty e^{-\frac{1}{2}\alpha^2} \sin \alpha x \, \phi(x)\, dx.$$

We cannot differentiate this equation by the ordinary rule, since

$$\int_0^\infty \frac{\partial}{\partial \alpha} (e^{-\frac{1}{2}\alpha^2} \sin \alpha x)\, \phi(x)\, dx$$

is divergent. But we can obtain the value of the integral on the right by a procedure devised by M. de la Vallée-Poussin.*

If $$\alpha > \beta > 0,$$

$$e^{-\frac{1}{2}\alpha^2} \sin \alpha x - e^{-\frac{1}{2}\beta^2} \sin \beta x = \int_\beta^\alpha \frac{\partial}{\partial \alpha} (e^{-\frac{1}{2}\alpha^2} \sin \alpha x)\, d\alpha$$

$$= \int_\beta^\alpha e^{-\frac{1}{2}\alpha^2} (x \cos \alpha x - \alpha \sin \alpha x)\, d\alpha,$$

and so

$$e^{-\frac{1}{2}\alpha^2} \psi(\alpha) - e^{-\frac{1}{2}\beta^2} \psi(\beta)$$

$$= \lim_{X=\infty} \int_0^X \phi(x)\, dx \int_\beta^\alpha e^{-\frac{1}{2}\alpha^2} (x \cos \alpha x - \alpha \sin \alpha x)\, d\alpha$$

$$= \lim_{X=\infty} \int_\beta^\alpha e^{-\frac{1}{2}\alpha^2} d\alpha \int_0^X (x \cos \alpha x - \alpha \sin \alpha x)\, \phi(x)\, dx.$$

Now

$$\int_0^X (x \cos \alpha x - \alpha \sin \alpha x)\, e^{\frac{1}{2}x^2} dx \int_x^\infty e^{-\frac{1}{2}t^2} dt$$

$$= \int_0^X \cos \alpha x \, \frac{d}{dx} (e^{\frac{1}{2}x^2})\, dx \int_x^\infty e^{-\frac{1}{2}t^2} dt$$

$$- \int_0^X \alpha \sin \alpha x \, e^{\frac{1}{2}x^2} dx \int_x^\infty e^{-\frac{1}{2}t^2} dt$$

$$= \left[\cos \alpha x \, e^{\frac{1}{2}x^2} \int_x^\infty e^{-\frac{1}{2}t^2} dt \right]_0^X + \int_0^X \cos \alpha x \, dx$$

* See Stolz, *Grundzüge*, III., p. 28,

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(on integrating by parts)

$$= \frac{\sin \alpha X}{\alpha} + e^{\frac{1}{2}X^2} \cos \alpha X \int_X^\infty e^{-\frac{1}{2}t^2} dt - \sqrt{(\tfrac{1}{2}\pi)}.$$

Now

$$\left| \int_\beta^\alpha e^{-\frac{1}{2}\alpha^2} d\alpha \left[e^{\frac{1}{2}X^2} \cos \alpha X \int_X^\infty e^{-\frac{1}{2}t^2} dt \right] \right|$$

$$< (\alpha - \beta) e^{\frac{1}{2}X^2} \int_X^\infty e^{-\frac{1}{2}t^2} dt,$$

the limit of which for $X = \infty$ is 0; and*

$$\lim_{X=\infty} \int_\beta^\alpha \frac{\sin \alpha X}{\alpha} e^{-\frac{1}{2}\alpha^2} d\alpha = 0.$$

Hence

$$e^{-\frac{1}{2}\alpha^2} \psi(\alpha) - e^{-\frac{1}{2}\beta^2} \psi(\beta) = -\sqrt{(\tfrac{1}{2}\pi)} \int_\beta^\alpha e^{-\frac{1}{2}\alpha^2} d\alpha.$$

Now $$e^{-\frac{1}{2}\beta^2} \psi(\beta) = \int_0^\infty e^{-\frac{1}{2}\beta^2} \sin \beta x \, \phi(x) \, dx,$$

and it follows from the fact that† for large values of x,

$$\phi(x) = \frac{1}{x} + \ldots,$$

that $$\lim_{\beta=0} e^{-\frac{1}{2}\beta^2} \psi(\beta) = \frac{\pi}{2}.$$

Hence

$$e^{-\frac{1}{2}\alpha^2} \psi(\alpha) = \frac{\pi}{2} - \sqrt{(\tfrac{1}{2}\pi)} \int_0^\alpha e^{-\frac{1}{2}\alpha^2} d\alpha = \sqrt{(\tfrac{1}{2}\pi)} \int_\alpha^\infty e^{-\frac{1}{2}\alpha^2} d\alpha,$$

i.e. $$\psi(\alpha) = \sqrt{\left(\frac{\pi}{2}\right)} e^{\frac{1}{2}\alpha^2} \int_\alpha^\infty e^{-\frac{1}{2}t^2} dt = \sqrt{\left(\frac{\pi}{2}\right)} \phi(\alpha).$$

Hence $$\phi(x) = e^{\frac{1}{2}x^2} \int_x^\infty e^{-\frac{1}{2}t^2} dt$$

is its own reciprocal of the second kind.

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* Kronecker, *Vorlesungen über Integrale*, p. 69.
† *Quarterly Journal*, Vol. XXXIV., p. 45.

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§ 6. Cauchy proved that if $\phi(x)$, $\psi(x)$ were reciprocal functions of the first kind, and α, β were real quantities such that $\alpha\beta = \pi$, then

$$\sqrt{\alpha}\{\tfrac{1}{2}\phi(0) + \phi(\alpha) + \phi(2\alpha) + \ldots\}$$
$$= \sqrt{\beta}\{\tfrac{1}{2}\psi(0) + \psi(\beta) + \psi(2\beta) + \ldots\},$$

of course under certain conditions. The integral formulæ, which I gave in § 1, can be deduced from a similar relation which holds between two reciprocal functions of the second kind.

Suppose that $\phi(x)$ is a function of the complex variable $x = \xi + i\eta$, analytic within and on the boundary of the rectangle formed by the lines

$$\xi = 0,\ \xi = N\pi,\ \eta = 0,\ \eta = k > 0,$$

and that $\phi(\xi + i\eta)$ tends steadily to the limit 0 when ξ tends to ∞, for any value of η in $(0, k)$. And let us integrate

$$\int \frac{\phi(\lambda x)}{\cos \pi x}\, dx \quad (\lambda > 0)$$

round this rectangle, avoiding the poles by small semicircles in the usual way. In the limit, when the radii of the semicircles are indefinitely diminished, and N is indefinitely increased, we obtain the equation

$$(6)\quad P\int_0^\infty \frac{\phi(\lambda\xi)\, d\xi}{\cos \pi\xi} + i\sum_0^\infty (-)^n \phi\{(n + \tfrac{1}{2})\lambda\}$$
$$- \int_0^\infty \frac{\phi\{\lambda(\xi + ik)\}}{\cos \pi(\xi + ik)}\, d\xi - i\int_0^k \frac{\phi(\lambda i\eta)}{\cosh \pi\eta}\, d\eta = 0.$$

When k is indefinitely diminished the last integral tends to 0.

I shall suppose now that the first two derivates of $\phi(x)$ are continuous throughout the rectangle and vanish for $\xi = \infty$, and that the integrals

$$\int_0^\infty \phi\{\lambda(\xi + ik)\}\, e^{(2n+1)\pi\xi i}\, d\xi,$$

$$\int_0^\infty \phi'\{\lambda(\xi + ik)\}\, e^{(2n+1)\pi\xi i}\, d\xi,$$

$$\int_0^\infty \phi''\{\lambda(\xi + ik)\}\, e^{(2n+1)\pi\xi i}\, d\xi$$

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are uniformly convergent for $n = 0, 1, 2, \ldots$, and all values of k in a small interval $(0, k')$. Then

$$\int_0^\infty \frac{\phi\{\lambda(\xi+ik)\}}{\cos\pi(\xi+ik)}\,d\xi$$

$$= 2\sum_0^\infty (-)^n e^{-(2n+1)\pi k}\int_0^\infty \phi\{\lambda(\xi+ik)\}\,e^{(2n+1)\pi\xi i}\,d\xi.$$

Now

$$\int_0^\infty \phi\{\lambda(\xi+ik)\}\,e^{(2n+1)\pi\xi i}\,d\xi$$

$$= \frac{1}{(2n+1)\pi i}\left\{-\phi(\lambda ik) - \lambda\int_0^\infty \phi'\{\lambda(\xi+ik)\}\,e^{(2n+1)\pi\xi i}\,d\xi\right\}$$

$$= -\frac{\phi(\lambda ik)}{(2n+1)\pi i}$$

$$+ \frac{\lambda}{(2n+1)^2\pi^2}\left\{-\phi'(\lambda ik) - \lambda\int_0^\infty \phi''\{\lambda(\xi+ik)\}\,e^{(2n+1)\pi\xi i}\,d\xi\right\}.$$

But $$\Sigma\frac{(-)^n}{2n+1}e^{-(2n+1)\pi k}$$

and $$\Sigma\frac{(-)^n}{(2n+1)^2}e^{-(2n+1)\pi k}$$

are continuous for $k = 0$.

And

$$\Sigma\frac{(-)^n}{(2n+1)^2}e^{-(2n+1)\pi k}\int_0^\infty \phi''\{\lambda(\xi+ik)\}\,e^{(2n+1)\pi\xi i}\,d\xi$$

is uniformly convergent in $(0, k')$, and therefore continuous for $k = 0$. Hence

$$\lim_{k=0}\int_0^\infty \frac{\phi\{\lambda(\xi+ik)\}}{\cos\pi(\xi+ik)}\,d\xi = 2\sum_0^\infty (-)^n\int_0^\infty \phi(\lambda\xi)\,e^{(2n+1)\pi\xi i}\,d\xi.$$

§ 7. Equating the real and imaginary parts in (6), and making $k = 0$, we obtain

$$(7)\quad P\int_0^\infty \frac{\phi(\lambda\xi)\,d\xi}{\cos\pi\xi} = 2\sum_0^\infty (-)^n\int_0^\infty \phi(\lambda\xi)\cos(2n+1)\pi\xi\,d\xi,$$

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and

$$(8)\quad \sum_0^\infty (-)^n \phi\{(n+\tfrac{1}{2})\lambda\} = 2\sum_0^\infty (-)^n \int_0^\infty \phi(\lambda\xi)\sin(2n+1)\pi\xi\, d\xi.$$

These formulæ may also be obtained by other* methods.

§ 8. If now $\psi(x)$ is the reciprocal of the second kind of $\phi(x)$,

$$\psi(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin \alpha x\, \phi(x)\, dx,$$

and $$\int_0^\infty \phi(\lambda\xi)\sin(2n+1)\pi\xi\, d\xi = \frac{1}{\lambda}\sqrt{\left(\frac{\pi}{2}\right)}\psi\left\{(2n+1)\frac{\pi}{\lambda}\right\}.$$

And if $\lambda\mu = 2\pi$,

$$(9)\quad \sqrt{\lambda}\sum_0^\infty (-)^n \phi\{(n+\tfrac{1}{2})\lambda\} = \sqrt{\mu}\sum_0^\infty (-)^n \psi\{(n+\tfrac{1}{2})\mu\}.$$

If, for instance,

$$\phi(x) = \frac{1}{x^s} \quad (0 < s < 1),$$

$$\psi(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin \alpha x}{x^s}\, dx$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\Gamma(1-s)\cos\tfrac{1}{2}s\pi}{\alpha^{1-s}},$$

and the formula gives, after a little reduction,

$$(10)\quad \sum_0^\infty \frac{(-)^n}{(2n+1)^s} = \left(\frac{2}{\pi}\right)^{1-s} \cos\tfrac{1}{2}s\pi\, \Gamma(1-s) \sum_0^\infty \frac{(-)^n}{(2n+1)^{1-s}}.$$

This formula was originally found by Schlömilch† from the theory of Fourier series, and has since been generalised by various writers, notably Lipschitz.‡ Schlömilch also showed how to deduce the general formula (9).

It is to be observed that in the particular case in which $\phi(x) = x^{-s}$, the argument of § 6 requires to be modified slightly, owing to the fact that $\phi(x)$ becomes infinite at $x = 0$.

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* See a paper 'On differentiation and integration of divergent series,' *Trans. Camb. Phil. Soc.*, XIX., p. 297.

† *Zeitschrift für Math. und Phys.*, iii., p. 130.

‡ *Crelle*, CV., p. 127.

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§ 9. Now

$$(11)\quad \int_0^\infty \frac{e^{-k^2x^2}}{1+x^2}\,dx = \sqrt{\pi}\,.e^{k^2}\int_k^\infty e^{-t^2}\,dt.$$

For if we write r for k^2, and differentiate with respect to r, we find that the integral is a solution of

$$\frac{du}{dr} - u = -\tfrac{1}{2}\sqrt{\frac{\pi}{r}},$$

the solution of which is

$$u = Ce^r - \tfrac{1}{2}\sqrt{\pi}\,\frac{1}{D-1}\left(\frac{1}{\sqrt{r}}\right),$$

$\left(\text{where } D \equiv \dfrac{d}{dr}\right)$ or

$$= \tfrac{1}{2}\sqrt{\pi}\,e^r \int_r^K \frac{e^{-r}}{\sqrt{r}}\,dr,$$

where K is an arbitrary constant. To determine K we notice that for $r=0$, $u=\frac{1}{2}\pi$, so that $K=\infty$.*

But

$$\int_0^\infty \frac{e^{-a^2x^2}}{\cosh \pi x}\,dx = \frac{4}{\pi}\int_0^\infty \sum_0^\infty \frac{(-)^n(2n+1)}{(2n+1)^2+4x^2}\,e^{-a^2x^2}\,dx$$

$$= \frac{2}{\pi}\sum_0^\infty (-)^n \int_0^\infty \frac{e^{-(n+\frac{1}{2})^2a^2u^2}}{1+u^2}\,du$$

$$= \frac{2}{\pi}\sum_0^\infty (-)^n F(n+\tfrac{1}{2})\,a,$$

say. And

$$F(x) = \sqrt{\pi}\,e^{x^2}\int_x^\infty e^{-t^2}\,dt,$$

so that $F\left(\dfrac{x}{\sqrt{2}}\right)$ is its own reciprocal of the second kind. Hence

$$\sqrt{\alpha}\,\Sigma\,(-)^n F\left\{(n+\tfrac{1}{2})\,\frac{\alpha}{\sqrt{2}}\right\} = \sqrt{\beta}\,\Sigma\,(-)^n F\left\{(n+\tfrac{1}{2})\,\frac{\beta}{\sqrt{2}}\right\},$$

* See G. F. Meyer's edition of Dirichlet's lectures.

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if $\alpha\beta = 2\pi$, or

$$\sqrt{\alpha}\,\Sigma\,(-)^n F\{(n+\tfrac{1}{2})\,\alpha\} = \sqrt{\beta}\,\Sigma\,(-)^n F\{(n+\tfrac{1}{2})\,\beta\},$$

if $\alpha\beta = \pi$. Thus, finally,

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-\alpha^2x^2}}{\cosh \pi x}\,dx = \sqrt{\beta}\int_0^\infty \frac{e^{-\beta^2x^2}}{\cosh \pi x}\,dx,$$

if $\alpha\beta = \pi$.

§ 10. We can also prove this without the use of the equation (9). For

$$\int_0^\infty \frac{e^{-\alpha^2\xi^2}}{\cosh \pi\xi}\,d\xi = 2\sum_0^\infty (-)^n \int_0^\infty e^{-\alpha^2\xi^2-(2n+1)\pi\xi}\,d\xi,$$

and since

$$\int_0^\infty e^{-\alpha^2\xi^2-2\lambda\xi}\,d\xi = e^{\left(\frac{\lambda}{\alpha}\right)^2}\int_{\frac{\lambda}{\alpha^2}}^\infty e^{-\alpha^2t^2}\,dt$$

$$= \frac{1}{\alpha}e^{\left(\frac{\lambda}{\alpha}\right)^2}\int_{\frac{\lambda}{\alpha}}^\infty e^{-u^2}\,du = \frac{1}{\alpha\sqrt{\pi}}F\left(\frac{\lambda}{\alpha}\right),$$

$$\int_0^\infty \frac{e^{-\alpha^2\xi^2}}{\cosh \pi\xi}\,d\xi = \frac{2}{\alpha\sqrt{\pi}}\sum_0^\infty (-)^n F\{(n+\tfrac{1}{2})\,\beta\},$$

where $\alpha\beta = \pi$; and therefore

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-\alpha^2\xi^2}}{\cosh \pi\xi}\,d\xi = \sqrt{\beta}\int_0^\infty \frac{e^{-\beta^2\xi^2}}{\cosh \pi\xi}\,d\xi.$$

§ 11. In the same way it may be shown that (4) of § 1 is equivalent to

$$\sqrt{\alpha}\{F(\tfrac{2}{3}\alpha) - F(\tfrac{4}{3}\alpha) + F(\tfrac{8}{3}\alpha) - F(\tfrac{10}{3}\alpha) + \ldots\}$$
$$= \sqrt{\beta}\{F(\tfrac{2}{3}\beta) - F(\tfrac{4}{3}\beta) + F(\tfrac{8}{3}\beta) - F(\tfrac{10}{3}\beta) + \ldots\}$$

($\alpha\beta = \frac{3}{4}\pi$), a formula which may be found as in § 6 by means of the contour integral

$$\int \frac{\phi(\alpha x)\sin \pi x}{\sin 3\pi x}\,dx.$$

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And (5) of § 1 is equivalent to

$$\sqrt{\alpha}\{F(\tfrac{1}{2}\alpha)+F(\tfrac{3}{2}\alpha)-F(\tfrac{5}{2}\alpha)-F(\tfrac{7}{2}\alpha)+\ldots\}$$
$$=\sqrt{\beta}\{F(\tfrac{1}{2}\beta)+F(\tfrac{3}{2}\beta)-F(\tfrac{5}{2}\beta)-F(\tfrac{7}{2}\beta)+\ldots\}$$

($\alpha\beta=\frac{1}{2}\pi$), which may be found by means of the integral

$$\int\frac{\phi(\alpha x)}{\cos\pi x}e^{-\frac{1}{2}\pi i x^2}dx.$$

§ 12. The formulæ (3), (4), and (5) of § 1, which were proved on the supposition that a was real, are easily extended to other values of a. For

$$a^{\frac{1}{4}}\int_0^\infty\frac{e^{-a\xi^2}d\xi}{\cosh\pi\xi}-b^{\frac{1}{4}}\int_0^\infty\frac{e^{-b\xi^2}d\xi}{\cosh\pi\xi}$$

is an analytic function of a throughout the region for which the real part of a is positive.

If we write $ae^{i\phi}$ $(a>0)$ for a, $be^{-i\phi}$ for b, where $ab=\pi^2$, and make ϕ increase from 0 to $\frac{1}{2}\pi$, we obtain ultimately

$$(12)\quad a^{\frac{1}{4}}e^{\frac{1}{8}\pi i}\int_0^\infty\frac{e^{-ai\xi^2}d\xi}{\cosh\pi\xi}-b^{\frac{1}{4}}e^{-\frac{1}{8}\pi i}\int_0^\infty\frac{e^{bi\xi^2}}{\cosh\pi\xi}d\xi=0$$

($a, b>0$, $ab=\pi^2$).

In the case in which a is the reciprocal of an odd integer n, the value of the first integral has been found by Kronecker.*

Integrate

$$\int\frac{e^{\frac{\pi i x^2}{n}}dx}{\cos\pi x}$$

round the contour bounded by

$$\xi=0,\ \xi=\tfrac{1}{2}n,\ \eta=Y,\ \eta=-Y,$$

avoiding $x=\frac{1}{2}n$ by a small semicircle. The contribution of the sides $\eta=\pm Y$ vanishes when Y is made infinite. For consider

$$\int_0^{\frac{1}{2}n}\exp.\left\{\frac{\pi i}{n}(\xi\pm iY)^2\right\}\sec\pi(\xi\pm iY)\,d\xi.$$

* *Crelle*, cv., p. 345.

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The modulus of the subject sf integration is

$$\frac{\sqrt{2}\exp.\left(\mp\frac{2\pi\xi Y}{n}\right)}{\sqrt{(\cosh 2\pi Y+\cos 2\pi\xi)}},$$

which is less than a constant H for all values of ξ in $(0, \frac{1}{2}n)$ and all large values of Y. Hence we can choose ϵ so that

$$\left|\int_{\frac{1}{2}n-\epsilon}^{\frac{1}{2}n}\right|<\sigma$$

for all large values of Y, however small be σ. And if $0\leqq\xi\leqq\frac{1}{2}n-\epsilon$ the modulus of the subject of integration is less than

$$K\exp.\left(-\frac{2\epsilon\pi Y}{n}\right),$$

where K is a constant; and this tends uniformly to 0 for $Y=\infty$.

The contribution of $\xi=0$ is

$$-\int_{-\infty}^{\infty}\frac{e^{-\frac{\pi i\eta^2}{n}}}{\cosh\pi\eta}\,id\eta=-2i\int_0^{\infty}\frac{e^{-\frac{\pi i\eta^2}{n}}}{\cosh\pi\eta}\,d\eta,$$

when Y is made infinite. That of $\xi=\frac{1}{2}n$ is

$$P\int_{-\infty}^{\infty}\frac{e^{\frac{\pi i}{n}\left(\frac{n}{2}+i\eta\right)^2}}{\cos\pi\left(\frac{n}{2}+i\eta\right)}\,id\eta$$

$$=(-)^{\frac{n+1}{2}}e^{\frac{n\pi i}{4}}P\int_{-\infty}^{\infty}e^{-\frac{\pi i\eta^2}{n}}\frac{e^{-\pi\eta}}{\sinh\pi\eta}\,d\eta$$

$$=(-)^{\frac{n-1}{2}}2e^{\frac{n\pi i}{4}}\int_0^{\infty}e^{-\frac{\pi i\eta^2}{n}}\,d\eta$$

$$=(-)^{\frac{n-1}{2}}e^{\frac{(n-1)\pi i}{4}}\sqrt{n}.$$

That of the small semicircle round $x=\frac{1}{2}n$ is

$$-(-)^{\frac{n+1}{2}}ie^{\frac{n\pi i}{4}}.$$

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Hence

$$(13)\quad \int_0^\infty \frac{e^{-\frac{\pi i\eta^2}{n}}\,d\eta}{\cosh \pi\eta} = \frac{1}{2i}\left\{(-)^{\frac{n-1}{2}} e^{\frac{(n-1)\pi i}{4}} \sqrt{n} - 2i \Sigma (-)^{k+1}\epsilon_k e^{(k+\frac{1}{2})^2 \frac{\pi i}{n}}\right\},$$

where the summation extends from $k=0$ to $k=\frac{1}{2}(n-1)$, and $\epsilon_k=\frac{1}{2}$ if $k=\frac{1}{2}(n-1)$, $=1$ otherwise.

Thus

$$(14)\quad \int_0^\infty \frac{\cos\frac{\pi\eta^2}{n}}{\cosh \pi\eta}\,d\eta = \tfrac{1}{2}(-)^{\frac{n-1}{2}} \sin\frac{(n-1)\pi}{4}\sqrt{n} - \Sigma (-)^{k+1}\epsilon_k \cos\left\{(k+\tfrac{1}{2})^2\frac{\pi}{n}\right\},$$

$$(15)\quad \int_0^\infty \frac{\sin\frac{\pi\eta^2}{n}}{\cosh \pi\eta}\,d\eta = \tfrac{1}{2}(-)^{\frac{n-1}{2}} \cos\frac{(n-1)\pi}{4}\sqrt{n} + \Sigma (-)^{k+1}\epsilon_k \sin\left\{(k+\tfrac{1}{2})^2\frac{\pi}{n}\right\}.$$

If $n=1$ we find

$$(16)\quad \int_0^\infty \frac{\cos\pi\eta^2}{\cosh\pi\eta}\,d\eta = \frac{1}{2\sqrt{2}},\quad \int_0^\infty \frac{\sin\pi\eta^2}{\cosh\pi\eta}\,d\eta = \tfrac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right),$$

$$(17)\quad \int_0^\infty \frac{e^{\pm\pi i\eta^2}}{\cosh\pi\eta}\,d\eta = \frac{1\pm i(\sqrt{2}-1)}{2\sqrt{2}};$$

and as

$$\frac{1+i(\sqrt{2}-1)}{1-i(\sqrt{2}-1)} = \frac{-2+2\sqrt{2}+2i(\sqrt{2}-1)}{4-2\sqrt{2}} = \frac{1+i}{\sqrt{2}} = e^{\frac{1}{4}\pi i},$$

this affords a verification of equation (12).

It is evident that from (12) and (13) we can deduce the values of

$$\int_0^\infty \frac{\cos n\pi\eta^2}{\cosh\pi\eta}\,d\eta,\quad \int_0^\infty \frac{\sin n\pi\eta^2}{\cosh\pi\eta}\,d\eta,$$

for all odd integral values of n.

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§ 13. If n is even and $=2m$, the argument is the same except that $x=\frac{n}{2}$ is not a pole, and the contribution of $\xi=\frac{n}{2}$ is

$$\int_{-\infty}^{\infty} \frac{e^{\frac{\pi i}{n}\left(\frac{n}{2}+i\eta\right)^2}}{\cos\pi\left(\frac{n}{2}+i\eta\right)}\, id\eta$$

$$= (-)^m ie^{\frac{n\pi i}{4}} \int_{-\infty}^{\infty} e^{-\frac{\pi i\eta^2}{n}} \frac{e^{-\pi\eta}}{\cosh\pi\eta}\, d\eta$$

$$= (-)^m 2ie^{\frac{n\pi i}{4}} \int_{0}^{\infty} e^{-\frac{\pi i\eta^2}{n}}\, d\eta$$

$$= (-)^m ie^{\frac{(n-1)\pi i}{4}} \sqrt{n}.$$

Therefore

$$(18)\quad \int_0^\infty \frac{e^{-\frac{\pi i\eta^2}{n}}\, d\eta}{\cosh\pi\eta} = \tfrac{1}{2}(-)^{\frac{1}{2}n} e^{\frac{(n-1)\pi i}{4}} \sqrt{n} - \sum_{k=0}^{\frac{1}{2}n-1} (-)^{k+1} e^{(k+\frac{1}{2})^2 \frac{\pi i}{n}},$$

$$(19)\quad \int_0^\infty \frac{\cos\frac{\pi\eta^2}{n}}{\cosh\pi\eta}\, d\eta = \tfrac{1}{2}(-)^{\frac{1}{2}n} \cos\frac{(n-1)\pi}{4} \sqrt{n} - \sum_{k=0}^{\frac{1}{2}n-1} (-)^{k+1} \cos\left\{(k+\tfrac{1}{2})^2 \frac{\pi}{n}\right\},$$

$$(20)\quad \int_0^\infty \frac{\sin\frac{\pi\eta^2}{n}}{\cosh\pi\eta}\, d\eta = -\tfrac{1}{2}(-)^{\frac{1}{2}n} \sin\frac{(n-1)\pi}{4} \sqrt{n} + \sum_{k=0}^{\frac{1}{2}n-1} (-)^{k+1} \sin\left\{(k+\tfrac{1}{2})^2 \frac{\pi}{n}\right\}.$$

Thus we can find the values of

$$\int_0^\infty \frac{\cos\frac{\pi\eta^2}{n}}{\cosh\pi\eta}\, d\eta, \quad \int_0^\infty \frac{\sin\frac{\pi\eta^2}{n}}{\cosh\pi\eta}\, d\eta,$$

$$\int_0^\infty \frac{\cos n\pi\eta^2}{\cosh\pi\eta}\, d\eta, \quad \int_0^\infty \frac{\sin n\pi\eta^2}{\cosh\pi\eta}\, d\eta,$$

for all integral values of n.

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ON THE CONDITIONS OF INTEGRABILITY OF A FUNCTION OF A REAL VARIABLE.

By E. W. Hobson.

THE necessary and sufficient conditions that a limited function $f(x)$ defined for the interval (a, b) of the real variable x, is integrable in the domain of the variable, have been reduced by W. H. Young* for the first time to a form which involves only a statement as to the properties of the set of all the points of discontinuity of the function. It is my object here to shew that the interesting and important results obtained by Mr. Young are deducible in a simple manner from Prof. Osgood's theorem† relating to the content of the limiting set of a sequence of non-dense closed sets of points.

Let $G_1, G_2, \ldots, G_n, \ldots$ be a sequence of non-dense closed sets of points, such that each set is contained in the next one of the sequence, and let G denote the limiting set, *i.e.* a set which contains every point which belongs to any set of the sequence, and no other points. Osgood's theorem is that if $I(G_n)$ denotes the content of G_n, then provided G is closed, the content of G is the limit, when n is increased indefinitely, of $I(G_n)$. In general, however G is unclosed, in that case let us denote any closed set which is part of G, by L; it will be shewn that the content of L cannot exceed the limit, when n is indefinitely increased, of $I(G_n)$.

Since L and G_n are both closed, their common divisor $D(L, G_n)$ is closed; by the common divisor is meant the set of all points which are common to L and G_n. Moreover, $D(L, G_n)$ cannot have a content greater than $I(G_n)$, since $D(L, G_n)$ is a part or the whole of G_n. It thus appears that the sequence of sets $D(L, G_1), D(L, G_2), \ldots, D(L, G_n) \ldots$ is such that the content of each set $D(L, G_n)$ does not exceed the content $I(G_n)$ of the corresponding G_n. Now it is clear that L is the limit of $D(L, G_n)$, hence by Osgood's theorem, since L is closed, its content is the limit of that of $D(L, G_n)$, and this clearly does not exceed the limit of $I(G_n)$.

It has thus been shewn that the content of any closed set whatever, which is part of G, does not exceed the limit, when n is indefinitely increased, of the content of G_n.

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* See his "Note on the condition of integrability of a function of a real variable," *Quarterly Journal*, Vol. XXXV.

† See *American Journal*, Vol. XIX., 1897.

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If G_1, G_2, ..., G_n, ... are the sets of those points at which $f(x)$ has discontinuities which are $\geq k_1$, $\geq k_2$, ..., where $k_1, k_2, \ldots, k_n, \ldots$ is a sequence of diminishing positive numbers which converges to zero, it is well known that for any point-wise discontinuous function all these sets are closed, and that the necessary and sufficient conditions for the integrability of the function is that all of them have content zero. If $I(G_n) = 0$, for every value of n, it follows that the limit of $I(G_n)$ is zero, and hence that the content of any closed set contained in G cannot exceed zero.

Moreover, if the content of every closed set contained in G is zero, the contents of G_1, G_2, ..., each of which is contained in G, are all zero. This proves W. H. Young's theorem that the necessary and sufficient conditions of integrability of $f(x)$ are included in the statement, that every closed set which is contained in the set G, of points of discontinuity of $f(x)$, has zero content.

I may remark that conversely it may be easily shewn that Osgood's theorem is at once deducible from Young's theorem, that no closed set contained in G has greater content than the limit of $I(G_n)$. For if G is itself closed, it follows that the content of G is not greater than the limit of $I(G_n)$; moreover, the content of G cannot be less than the limit of $I(G_n)$, since for every value of n, $I(G) \geq I(G_n)$. It thus appears that $I(G)$ is equal to the limit of $I(G_n)$ when n is indefinitely increased.

A GENERAL DYNAMICAL THEOREM, AND ITS APPLICATION TO THE KINETIC THEORY OF GASES.

By J. H. Jeans, M.A. Trinity College, Cambridge.

Introduction.

§ 1. In a recently published paper, "The Kinetic Theory of Gases developed from a new standpoint,"* I shewed how the Kinetic Theory could be based, independently of special physical assumptions, upon the results of general abstract dynamics. The paper, however, applied only to a gas in which the conservation of energy was supposed to be satisfied throughout, and it was found necessary to end the

* *Phil. Mag.* [6], Vol. v., p. 597.

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paper by an admission that this supposition had not led to a satisfactory conclusion. It appeared that to obtain results in agreement with experiment the dissipation of energy must be taken into account throughout.

It might be thought at first sight, by those who have read the former paper, that it would be an easy matter to rewrite the paper, taking account of the dissipation of energy represented by an unknown general dissipation function. Unfortunately this is not so. In the former case the ultimate result happened to be entirely independent of the constitution of the molecules of the gas. In the present more general case, in which the gas is not supposed to satisfy the conservation of energy, no ultimate general result of this kind can be obtained. The objects of the present paper, and of the one immediately following it, are therefore first to develop the general dynamical theory as far as possible on the lines followed in the former paper, and then, when further progress is found to be impossible on account of our ignorance of the structure of the molecules, to examine a single illustrative case of the general theory, and attempt to deduce results for the molecules of nature.

§ 2. As a preliminary to the general dynamical theory, it must be explained how it is that a gas in nature is adequately represented by a system of similar molecules and a dissipation function, and we must investigate the appropriate form for this dissipation function.

We shall assume that the complete dynamical system consists of the molecules of the gas and the ether by which these molecules are surrounded, and we shall therefore suppose that all energy lost to the molecules is energy gained to the ether. But in estimating this quantity there is a difficulty, for we do not know sufficient about the line of demarcation between ether and matter to be able to say at what stage precisely the energy is transferred from the one to the other.

Happily there is no need to probe into this difficult question. The dynamics of those parts of the ether which do not contain any matter are, at any rate if we assume Maswell's electrodynamical theory, sufficiently well understood. If S_1, S_2 are small surfaces surrounding the molecules, we know that at any point outside these surfaces we have, if $\mathbf{E}$, $\mathbf{H}$ are the electric and magnetic forces,

$$\text{div. } \mathbf{E} = 0, \qquad \text{div. } \mathbf{H} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

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The total energy, electrostatic and electromagnetic, inside any one of the surfaces, say S_1, is known to be

$$\frac{1}{8\pi}\iiint (P^2 + Q^2 + R^2 + \alpha^2 + \beta^2 + \gamma^2)\, dx\, dy\, dz \quad \ldots\ldots(2),$$

where P, Q, R and α, β, γ are the components of electric and magnetic force respectively. The total flow of energy outwards across the surface S_1 is known, as shewn by Poynting, to be

$$\frac{1}{4\pi}\iint [\mathbf{EH}]\, dS_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Now if ξ_1, ξ_2, ... are the generalised coordinates of position and velocity of the system inside S_1 we have at every point inside S_1

$$\left.\begin{aligned} \text{div. } \mathbf{E} &= f(\xi_1, \xi_2, \ldots) \\ \text{div. } \mathbf{H} &= \phi(\xi_1, \xi_2 \ldots) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where f and ϕ are functions of the coordinates which are theoretically calculable, the latter being almost certainly zero. From equations 1 and 4 we can calculate $\mathbf{E}$ and $\mathbf{H}$ as functions of all the coordinates of the various systems, and from these we can calculate the Poynting flux given by expression (3), also as a function of these coordinates.

Let us, then, regard the energy of the system inside S_1 as the total material energy of this system, if any, plus the total electric energy, electrostatic and electromagnetic, associated with the ether inside S_1, this latter being given by expression (2). Then we may legitimately regard expression (3) as a dissipation function of the system inside S_1, and the motion of the system will be deducible from a knowledge of the forms of the energy function and of the dissipation function just defined. Taking this view of the system, the interaction between matter and ether is represented by the existence of the dissipation function, which may be interpreted physically as evidence of the radiation of electromagnetic waves and "pulses" into the ether; and the interaction of the various molecules *inter se* is represented by the fact that the energy and dissipation functions of system 1 depend not only on the coordinates of system 1, but also on those of systems 2, 3, ...

If, during the motion of the molecules, two or more of the surfaces S_1, S_2 ... intersect, an obvious slight modification

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must be made in the conventions just stated. Instead of integrating expression (3) over the surfaces separately to obtain the dissipation function, we must integrate only over the single closed surface formed by such as intersect. Similarly, instead of integrating expression (2) throughout the volumes enclosed by the surfaces separately to obtain the energy, we integrate only throughout the single connected volume formed by such as intersect. In this way we fail to obtain the energy and dissipation functions as the sums of contributions from individual molecules, and obtain these functions only in the aggregate, a limitation which is of no importance as regards the main results to be obtained. If, on the other hand, we are willing to assume that there is a definite fixed line of demarcation between matter and ether, and that two portions of matter cannot occupy the same space, then these difficulties are obviated, and we can associate definite terms of the energy and dissipation functions with each molecule.

The General Dynamical Theory.

§ 3. We can now, in the main, follow the method adopted in the former paper. That method being based on Liouville's Theorem, we must begin the present discussion by investigating the more general theorem by which this is to be replaced when the conservation of energy does not hold.

We consider a general dynamical system specified by the n coordinates $q_1, q_2 \ldots q_n$ and the n corresponding momenta $p_1, p_2, \ldots, p_n$. We can suppose all possible configurations of this system represented in a space of $2n$ dimensions, corresponding to the $2n$ variables

$$p_1, p_2, \ldots p_n, q_1, q_2, \ldots, q_n \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

If E is the energy of the system expressed as a function of the q's and $\dot{q}$'s, we have

$$p_s = \frac{\partial E}{\partial \dot{q}_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and the equations of motion of the system will be

$$\frac{\partial q_s}{\partial t} = \frac{\partial E}{\partial p_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$\frac{\partial p_s}{\partial t} = -\frac{\partial E}{\partial q_s} - \frac{\partial F}{\partial \dot{q}_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

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where E is now expressed as a function of

$$p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n,$$

and F is the dissipation function, expressed as a function of $q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n$. If, then, a point moves in the generalised space with the velocities prescribed by equations (7) and (8), its motion will represent the changes in the configuration of the original system, started with coordinates and velocities reprcsented by the starting point of the point in question.

As before, we do not consider the motion of a single point in this space, but the motion of an infinite number of points starting from all parts of the space. We therefore find it convenient to suppose the space filled with a continuous fluid, the fluid being the aggregate of all these points. Equations (7) and (8) are therefore the equations of motion of this fluid, and the elimination of the time from these equations will lead to the equations of the paths or stream lines along which the fluid moves.

The hydrodynamical equation of continuity in the generalised space, expressing the permanence of the points of which the fluid is composed, is

$$\frac{D\rho}{Dt} + \rho \sum_{s=1}^{s=n} \left(\frac{\partial \dot{p}_s}{\partial p_s} + \frac{\partial \dot{q}_s}{\partial q_s} \right) = 0 \quad \ldots\ldots\ldots\ldots (9),$$

where, as usual, $D\rho/Dt$ denotes differentiation following an element of fluid in its motion.

From equations (7) and (8), we obtain

$$\frac{\partial \dot{p}_s}{\partial p_s} + \frac{\partial \dot{q}_s}{\partial q_s} = - \frac{\partial^2 F}{\partial p_s \partial \dot{q}_s},$$

so that equation (9) becomes

$$\frac{D\rho}{Dt} = \rho \sum_{s=1}^{s=n} \frac{\partial^2 F}{\partial p_s \partial \dot{q}_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

In the special case in which $F = 0$, this reduces to

$$D\rho/Dt = 0,$$

which is, of course, the expression of Liouville's Theorem. Equation (10) therefore expresses the general theorem which must replace that of Liouville when the conservation of energy does not hold.

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§ 4. In the former case, in which we had $F=0$, the procedure was one of extreme simplicity. Since we had $D\rho/Dt=0$, it was clear that a distribution of fluid which was of uniform density at one instant would remain of uniform density throughout all time. Hence a generalised space filled with fluid of uniform density supplied a basis of investigation which was independent of the time. It was therefore possible to prove, by taking a census of the different points in the generalised space, that all except an infinitesimal fraction of the space represented systems for which the well-known propositions of the Kinetic Theory—equipartition of energy, etc.—were true, and hence that these propositions would be true (except for a theoretical infinitesimal probability of error) for a gas selected at random.

In the present instance this method fails because the space filled initially with homogenous fluid does not supply a basis of calculation which is independent of the time. It may be (and, as will be seen, actually is) the case that the fluid tends to crowd just to those parts of the space which represent systems in which the former propositions are not true. To investigate these questions, we commence by seeking for a basis which shall be independent of the time.

§ 5. For the sake of brevity, let us write

$$\sum_{s=1}^{s=n} \frac{\partial^2 F}{\partial p_s \partial \dot{q}_s} = \Delta F \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

so that we can write equation (10) in the form

$$\frac{D\rho}{Dt} = \rho \Delta F \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

We now have

$$\begin{aligned} \Delta F &= \sum_{s=1}^{s=n} \frac{\partial^2 F}{\partial p_s \partial \dot{q}_s} \\ &= \sum_{s=1}^{s=n} \sum_{t=1}^{t=n} \frac{\partial^2 F}{\partial p_s \partial p_t} \frac{\partial p_t}{\partial \dot{q}_s}, \\ &= \sum_{s=1}^{s=n} \sum_{t=1}^{t=n} \frac{\partial^2 F}{\partial p_s \partial p_t} \frac{\partial^2 E}{\partial \dot{q}_s \partial \dot{q}_t} \end{aligned}$$

by equation (6), expressing ΔF as a function of the q's only. Since E and F are both positive for all values of the variables,

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it follows that ΔF is positive throughout the space, and hence from equation (12) it is clear that as we follow the motion of any element ρ continually increases. Also, as we follow the motion E decreases, so that the motion of the points in our imaginary space is one of concentration about minimum values of E, and, in particular, about $E=0$.

§ 6. The change in E as we follow any element in its motion is of course

$$\frac{DE}{Dt} = \sum_{s=1}^{s=n} \left(\frac{\partial E}{\partial q_s} \frac{\partial q_s}{\partial t} + \frac{\partial E}{\partial p_s} \frac{\partial p_s}{\partial t} \right),$$

$$= - \sum_{s=1}^{s=n} \frac{\partial E}{\partial p_s} \frac{\partial F}{\partial \dot{q}_s}.$$

by equations (6) and (7),

$$= - \sum_{s=1}^{s=n} \frac{\partial E}{\partial \dot{q}_s} \frac{\partial F}{\partial p_s},$$

$$= - \sum_{s=1}^{s=n} p_s \frac{\partial F}{\partial p_s},$$

$$= - 2F \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

From equations (12) and (13) it appears that as we follow the motion of any element

$$\frac{D\rho}{DE} = - \rho \frac{\Delta F}{2F} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

and D may now be regarded as the symbol of differentiation along a stream-line in the imaginary space. The equation just obtained has the integral

$$\rho = \rho_0 e^{-\int \frac{\Delta F}{2F} dE} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (15),$$

where the integral is taken along any stream-line, and ρ_0 is the value of ρ at the lower limit of integration.

§ 7. Equation (15) is an equation between ρ and the coordinates in the imaginary space: it does not involve the time. Hence it follows that, provided the initial distribution of points in the space is such as to satisfy equation (15) at every point, the motion of these points will be "steady

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motion" in the hydrodynamical sense. To obtain such a motion, then, we may assign arbitrary densities over any one of the loci $E=$ constant; the density of distribution of points is then uniquely determined throughout the whole space, and is given by equation (15). The space, filled with fluid in this way, supplies what we require, namely a basis of investigation which is independent of the time.

By distributing the arbitrary densities in different ways, we shall obtain the various distributions of points in the imaginary space which are appropriate to the discussion of different problems—*i.e.* to the discussion of gases about which various data are given. For instance, suppose we know that a gaseous substance has cooled, undisturbed by external agencies, from some higher temperature at which its energy was E_0, and that nothing further as to its past history is known, then we may appropriately suppose the imaginary space filled with a distribution of points such that the density over $E=E_0$ is constant, say equal to unity. If the present energy is E_1, the density of points over $E=E_1$ is, by equation (15),

$$\rho = \exp.\left(\int_{E_1}^{E_0} \frac{\Delta F}{2F}\, dE\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (16).$$

Application to a system with an infinite number of degrees of freedom.

§ 8. We now introduce the simplifying assumption that the system consists of a very great number of parts (*e.g.* molecules), so that n the number of momenta is very great. Let $\overline{p}^2$ denote the mean value of $p_1^2, p_2^2, \ldots, p_n^2$ at any point in the imaginary space. Then in general E is of the same order of magnitude as $n\overline{p}^2$, while $\Delta F/2F$ is obviously of the order of $1/\overline{p}^2$. Hence we can write

$$\frac{1}{n}\int_{E_1}^{E_0} \frac{\Delta F}{2F}\, dE = \Phi \quad \ldots\ldots\ldots\ldots\ldots\ldots (17),$$

and Φ remains finite when n becomes infinite, being, as we have seen, always positive.

The distribution of density in the region $E=E_1$ is now, by equation (16),

$$\rho = e^{n\Phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18).$$

There is therefore a point (or series of points) of maximum

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density obtained by making Φ a maximum, and the density here is infinitely greater than that elsewhere. We have now to examine in what way the density falls off as we recede from this point, so as to determine whether the whole aggregate of fluid for which $E = E_1$ may be regarded as being concentrated about this point (or series of points).

Let Φ_0 denote a maximum value of Φ, and suppose that this value occurs at the point $p'_1, p'_2, \ldots, q'_n$; then in the immediate neighbourhood of this point we can expand Φ in the form

$$\Phi = \Phi_0 + (p_1 - p'_1)\frac{\partial \Phi}{\partial p_1} + \ldots,$$

$$+ \tfrac{1}{2}(p_1 - p'_1)^2 \frac{\partial^2 \Phi}{\partial p_1^2} + \ldots \quad \ldots\ldots\ldots\ldots (19),$$

where the differential coefficients are all evaluated at the point at which $\Phi = \Phi_0$. Since Φ is, by hypothesis, a maximum so long as E is kept constant, it follows that over the surface $E = E_1$,

$$(p_1 - p'_1)\frac{\partial \Phi}{\partial p_1} + \ldots + (q_n - q'_n)\frac{\partial \Phi}{\partial q_n} = 0 \quad \ldots\ldots (20).$$

Hence the region for which $E = E_1$, and for which Φ lies between Φ_0 and an adjacent value Φ_1, is bounded by

$$\tfrac{1}{2}(p_1 - p'_1)^2 \frac{\partial^2 \Phi}{\partial p_1^2} + \ldots = \Phi_0 - \Phi_1 \quad \ldots\ldots\ldots (21).$$

When $(p_1 - p'_1)^2, \ldots$, are small enough, this is the equation of a small closed boundary, an "ellipsoid" of $2n$ dimensions. The volume of this ellipsoid is a Dirichlet integral, of which the value is found to be

$$\frac{(2\pi)^{\frac{1}{2}n}}{\Gamma(1 + \frac{1}{2}n)} (\Phi_0 - \Phi_1)^{\frac{1}{2}n} \Delta^{-\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots (22),$$

where

$$\Delta = \begin{vmatrix} \frac{\partial^2 \Phi}{\partial p_1^2}, & \frac{\partial^2 \Phi}{\partial p_1 \partial p_2}, & \ldots \\ \frac{\partial^2 \Phi}{\partial p_1 \partial p_2}, & \frac{\partial^2 \Phi}{\partial p_2^2}, & \ldots \\ \ldots\ldots\ldots\ldots & & \end{vmatrix} \ldots\ldots\ldots\ldots\ldots (23).$$

The equation of the plane $E = E_1$ is equation (20), hence the

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maximum normal distance ϖ of the ellipsoid from this plane is found to be given by

$$\varpi^2 = \frac{\left(\frac{\partial\Phi}{\partial p_1}\right)^2 \Big/ \left(\frac{\partial^2\Phi}{\partial p_1^2}\right)^2 + \ldots}{\left(\frac{\partial\Phi}{\partial p_1}\right)^2 + \ldots} \quad (\Phi_0 - \Phi_1) \ldots\ldots\ldots (24).$$

The intersection of the plane (20) with the $2n$ dimensional ellipsoid (21) determines a $(2n-1)$ dimensional ellipsoid. This latter ellipsoid contains all points for which

$$E = E_1 \text{ and } \Phi_1 < \Phi < \Phi_0.$$

If Ω be its volume, the volume of the original ellipsoid (21) is proportional to $\Omega\varpi$, and is equal to expression (22). Hence we see that the volume Ω may be written in the form

$$\Omega = C\,(\Phi_0 - \Phi_1)^{\frac{1}{2}(n-1)},$$

where C does not involve Φ_1, being given by

$$C = \frac{(2\pi)^{\frac{1}{2}n}\Delta^{-\frac{1}{2}}}{\Gamma\left(1+\frac{1}{2}n\right)} \left[\frac{\left(\frac{\partial\Phi}{\partial p_1}\right)^2 + \ldots}{\left(\frac{\partial\Phi}{\partial p_1}\right)^2 \Big/ \left(\frac{\partial^2\Phi}{\partial p_1^2}\right)^2 + \ldots}\right]^{\frac{1}{2}} \ldots\ldots\ldots (25).$$

By differentiation, the volume for which $E = E_1$, while Φ lies between Φ_1 and $\Phi_1 + d\Phi$, will be $\frac{1}{2}(n-1)\,C(\Phi_0 - \Phi_1)^{\frac{1}{2}(n-3)}$. Hence the aggregate mass of fluid representing systems for which $E = E_1$, and Φ lies between Φ_0 and Φ_1, is

$$\tfrac{1}{2}(n-1)\,C\int_{\Phi_1}^{\Phi_0} e^{n\Phi}\,(\Phi_0 - \Phi)^{\frac{1}{2}(n-3)}\,d\Phi \quad \ldots\ldots\ldots (26).$$

The integrand vanishes at $\Phi = \Phi_0$. After this, as Φ decreases, the integrand increases until it reaches its only maximum, which occurs when $\Phi_0 - \Phi = \frac{1}{2}\left(1 - \frac{3}{n}\right)$, and then decreases. In the limit when $n = \infty$, this maximum is given by

$$\Phi = \Phi_0 - \tfrac{1}{2} \quad \ldots\ldots\ldots (27),$$

and if this value lies between Φ_0 and Φ_1, the whole value of the integral (26) is given by contributions from the immediate neighbourhood of this maximum, and is equal to

$$C\Gamma\left\{\tfrac{1}{2}(n+1)\right\} n^{-\frac{1}{2}(n-1)}\, e^{n\Phi_0} \ldots\ldots\ldots (28).$$

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This result, however, only holds subject to the condition that we may neglect terms of order higher than the second in equation (19).

§ 9. Assuming for the moment that this condition is satisfied, our result shows that all the fluid for which $E = E_1$, and which is in the neighbourhood of the point of maximum density $\Phi = \Phi_0$, may be supposed to lie within the boundary defined by equation (19). In other words, the fluid is seen to flow into small clusters of stream-lines, a cluster surrounding each maximum value of Φ. All except an infinitesimally small proportion of the fluid for which $E = E_1$ will lie in one or other of these clusters.

The amount of fluid surrounding the maximum $\Phi = \Phi_0$ is given by expression (28). In comparing the amounts surrounding two different maxima Φ_0 and Φ_1, the ratio of the two factors $e^{n\Phi_0}$, $e^{n\Phi'_0}$ will of course be infinite or zero according as $\Phi_0 - \Phi'_0$ is positive or negative, but it is unfortunately impossible to determine the ratio of the two C's, given by equation (25), for the ratio of the two Δ's (equation 23) may be infinite or zero. We therefore see that in general the ratio of two expressions of the form (28) will be either infinite or zero, but it is impossible to determine which. In other words, we can see that all except an infinitesimally small amount to the fluid will be in a single cluster, but it is impossible to identify the particular cluster.

§ 10. An exception to the statement that all the fluid will be in a single cluster occurs when there are a number of clusters which are similar in every respect, so that expression (28) is identical for all. This will, for instance, arise in the case of a gas, owing to the possibility of interchanging the *rôles* of the different molecules.

§ 11. We now revert to equations (19) and (21) and examine whether or not the condition is satisfied that we may neglect terms of degree higher than the second in $(p_1 - p_1')$, etc. Upon the assumption that we may do so we have found the value of $\Phi_0 - \Phi$ at the boundary of a cluster to be equal to $\frac{1}{2}$, (equation 27). Let us suppose that the corresponding value of $p_1 - p'_1$ is of the order of magnitude δp, then the first term which has been neglected is

$$(p_1 - p_1')^3 \frac{\partial^3 \Phi}{\partial p_1^3} \text{ or } \delta p^3 \frac{\partial^3 \Phi}{\partial p_1^3}.$$

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Failing definite knowledge, it is natural to suppose that $\frac{\partial^2\Phi}{\partial p_1^2}$ is of the order of magnitude of $\frac{\frac{1}{2}}{\delta p^2}$, and hence that the neglected term is of the same order of magnitude as $\frac{1}{2}$. In other words, the neglect of the remaining terms in equation (21) cannot be justified.

But this only affects the detail of the result we obtain: the main result remains. The value Φ_0 being still a maximum for Φ, the locus $\Phi = \Phi_1$ will surround the point Φ_0, although it will only be ellipsoidal when $\Phi_0 - \Phi_1$ is very small. The integral in expression (26) must be replaced by an integral of the form

$$\int_{\Phi_1}^{\Phi_0} e^{n\Phi} f(\Phi_0 - \Phi)\, d\Phi$$

in which $f(x)$ is an unknown, but always positive, function of x which become identical with $x^{\frac{1}{2}(n-3)}$ in the neighbourhood of $x = 0$. The integrand in this integral will, as before, have a maximum, and the integral will derive its whole value from the contributions supplied by values of Φ in the neighbourhood of this maximum. Also, as before, the value of the integral involves Φ_0 merely through the factor $e^{n\Phi_0}$. Again, then, it appears that the fluid will flow into clusters surrounding maximum values of Φ, and the strength of such a cluster will be proportional jointly to $e^{n\Phi}$ and to another factor. All except an infinitesimal fraction of the fluid will be in a single cluster, or, in cases such as that contemplated in § 10, family of similar clusters; but, as already explained, it is impossible to identify this cluster or family of clusters by general arguments of the kind which have so far been used.

§ 12. Under the special assumption that we may neglect terms of degree higher than the second in equation (19), we have at any point *inside* a cluster (equation 21)

$$(p_1 - p'_1)^2 \frac{\partial^2\Phi}{\partial p_1^2} + 2(p_1 - p'_1)(p_2 - p'_2)\frac{\partial^2\Phi}{\partial p_1 \partial p_2} + \ldots < 2(\Phi_0 - \Phi_1) < 1,$$

by equation (27).

In the most general case it is obvious that the quadratic expression on the left hand is finite. The expression is, however, by hypothesis, positive for all values of $(p_1 - p'_1)$, ..., and as it contains the $2n$ variables $p'_1 p'_2, \ldots, q'_n$, it may be expressed as a sum of $2n$ squares of linear functions of these variables. When n is infinite each square will therefore be

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vanishingly small, so that $p_1 - p'_1, p_2 - p'_2, \ldots$, will be infinitesimal. We may say, then, that inside any cluster the coordinates $p'_1, p'_2, \ldots$, of any point are indistinguishable from the coordinates $p_1, p_2, \ldots$, of the corresponding point on the central line of the cluster.

§ 13. The analysis has supposed the system to have started from an initial value of the energy $E = E_0$, and has, moreover, supposed all configurations for which $E = E_0$ to have been equally probable as the starting point. In nature it is not easy to find any value of E_0 which shall represent the conditions of the problem. If, for instance, our system is a gas, we may probably assert with confidence that the gas has cooled from some higher temparature at which its energy had some value E_0, but we cannot say that at this temperature all configurations for which $E = E_0$ were equally likely. For obviously if it had previously cooled from some still higher temperature this statement would not be true. Again the value $E_0 = \infty$ does not supply a solution of the difficulty, for in taking this value we should not only have to suppose that the gas had cooled from a temperature in which it probably could not exist as a gas, but we should also have to suppose that it had cooled *undisturbed* throughout infinite time.

§ 14. An escape from this difficulty is found as soon as we examine the way in which our work has depended on E_0. It will be seen that E_0 has entered only as a limit for integration in the definition of Φ, namely (from equation 17),

$$\Phi = \frac{1}{n}\int_{E_1}^{E_0} \frac{\Delta F}{2F}\, dE = -\frac{1}{n}\int_{E_1}^{E_0} \Delta F\, dt \ldots\ldots\ldots (29).$$

In this equation the integral has to be taken tentatively along all stream lines starting from $E = E_0$ and the course of the cluster or family is determined by the maximum values of Φ. The integrand contains a factor F in the denominator. From its nature F cannot be negative and in general will not vanish. If at a special point, F does vanish, then the integrand in the first integral becomes infinite, but the value of the integral taken over any part of the path which includes this point remains finite, because so long as $F = 0$ we have also $dE = 0$. The same result is of course obvious from the second form of the integral. If, however, F remains very small over a finite range of values of E, the contribution to the integral from this range of values will be very great. In

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this case the definition of Φ, for values of E below a finite range of small values of F, becomes approximately independent of the upper limit of integration, so that the stream lines on which Φ is a maximum become identical for all such upper limits. Thus so soon as a value of the energy is reached so small that it is possible for F to remain very small over a large range of values of E, we see that the positions of the clusters tend to become independent of their starting point. In other words, all clusters tend to combine into a single cluster; or again, all stream-lines tend to combine into a single stream-line.

The values of the energy at which the condition just stated is satisfied are the only values which are of much importance in nature. They correspond to temperatures at which a substance can remain in appreciably the same state for a considerable interval of time; and in considering this question it must be borne in mind that the unit of time appropriate for comparison is the interval between successive collisions of the same molecule—say 10^{-10} seconds. For obvious reasons, the state of a system which changes appreciably in 10^{-10} seconds is not of much interest to us.

When the condition in question is satisfied the system may be said to be in an "approximately steady state." It is then infinitely probable that it will be represented by a point on the single cluster (or family of clusters) just discussed. In this case it may be said to be in the "normal state," this state being analogous to the normal state previously found for a conservative gas.

Application to Statistical Mechanics.

§ 15. It has already been assumed that the system possesses an infinite number of degrees of freedom. The application of the results must therefore be solely to "statistical mechanics," and not to dynamical problems of the ordinary type. A dynamical problem may, in accordance with accepted usage, be said to be one of statistical mechanics when the data and objects of inquiry are not the actual values of the various coordinates, but the law of distribution of these coordinates. Since the data of a problem in statistical mechanics do not completely specify the dynamical coordinates of the system, we are, in a problem of statistical mechanics, discussing an infinite number of different systems at once, and without differentiation *inter se*, and the motion of these systems will naturally diverge in the course of time. It may be that after the motion a

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single statistical specification can be given which covers all except an infinitesimal fraction of the systems. In this case a solution may be said to have been found to the problem. It cannot be that a solution can be given which covers all the systems; the reason for this (*not* the formal proof of it) is that even after the initial system has been fully specified statistically there are still an infinite number of undetermined variables, and, by giving suitable values to these, we can obtain any chosen infinite number of relations between the coordinates of the final result, and can therefore cause this final result to disagree with any single statistical specification. It is therefore clear that a statistical problem must always have an element of uncertainty in its final solution, although in virtue of the infinite number of the variables, this uncertainty may be represented by an infinitesimally small probability of error: it may, in the terminology previously used, be "infinitely probable" that the result is true.

§ 16. This view of statistical mechanics is in accordance with the development of the former paper already referred to, and also with the development of the present paper. Leaving out of mention the infinitesimal probability of error, it being understood that this is inherent to any problem of statistical mechanics, it may be said that we have found that of our non-conservative systems, started from unknown configurations, all will, by the time a sufficiently low value of the energy has been reached, be represented in our general space by points on a single cluster or family of clusters.

In the case of a gas it is obvious that we must take the second alternative and say "family of clusters" rather than "single cluster." For, as has already been pointed out in § 10, by interchanging the *rôles* of any two molecules we get two distinct clusters, which are identical in so far as the elements which are of importance in the dynamical problem are concerned. In other ways, too, it is clear that we shall get further distinct clusters which are identical as regards the important elements of the problem. Let M be the number of clusters in any such family, and S the strength of each, given by an expression of the form of (28). Then the strength of the family of clusters is MS. Comparing two families of clusters, we have already seen that the ratio of the two S's will be infinite, so that whether or not the ratio of the two M's is infinite, the ratio of the MS's is, in general, infinite. Thus all the fluid belongs to a single family. It is clear that systems in all the M clusters have the same statistical specification.

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Hence we may sum up as follows:

Out of a number of systems which have a given value for the energy, this value being sufficiently low, and which have had their energy reduced from higher values by dissipation, the systems meanwhile being unaffected by external disturbance, all except an infinitesimal fraction will have the same statistical specification.

For a gas, then, we may sum up as follows. If nothing is known about a gas except that it has cooled undisturbed from some higher temperature, and is now at such a temperature that its changes of state are not very rapid (by "very rapid" being meant comparable with 10^{-10} seconds), then it is infinitely probable that it will have a definite statistical specification, which depends solely on its present energy.

The way in which this specification has, in theory, been arrived at shows how profoundly different it must be from that arrived at on the assumption of conservation of energy, namely, that which leads to equipartition.

ON THE PARTITION OF ENERGY IN A SYSTEM OF LOADED SPHERES.

By J. H. Jeans, M.A., Trinity College, Cambridge.

Introduction.

§ 1. THIS paper is intended as an illustration of the general dynamical theory developed in the paper immediately preceding. The precise connection between the two papers can be explained in a few words.

The preceding paper was, primarily, an attempt at an extension of the general dynamical theory of gases to the case of a non-conservative gas. The general dynamical theory of a conservative gas indicates that after a sufficient time a "normal state" will be reached in which the coordinates will be arranged according to a definite statistical law. This general theory can be illustrated by particular examples, and of these the simplest is that in which the molecules are regarded as elastic spheres. In this case the spheres can, so to speak, be followed throughout their motion, and it is found, as the result of the analysis due to Maxwell, Boltzmann, and

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Lorentz, and given, for example, in Boltzmann's *Vorlesungen über gastheorie*, that the result can in this way be obtained which is predicted by the general dynamical theory of a conservative gas.

The present paper aims at supplying a similar illustration and confirmation of the general theory of a non-conservative gas. We attempt to find a sufficiently simple non-conservative gas, trace the changes which occur during the motion, and examine whether the results are in accordance with those predicted in the preceding paper.

The ideal gas chosen for discussion is, as before, one in which the molecules are elastic spheres, but we shall now suppose that the spheres are "loaded," *i.e.* that their centre of gravity does not coincide with their geometrical centre. It is now necessary to take the rotation of the molecules into account, this not being necessary in the simpler case, as the collisions were not then capable of influencing the rotation in any way. In order to make the present analysis illustrate the theory of the last paper we shall ultimately suppose the rotation to be subject to dissipation, the retardation being proportional to the angular velocity. An indication of the method to be followed has been given in a paper published some time ago*; in the present paper the calculations will be given in detail.

§ 2. Let us suppose the distance of the centre of gravity from the geometrical centre in each molecule to be r. In the final result it is obvious that only even powers of r can occur, for we can replace r by $-r$ without altering the nature of the gas. The solution of the whole problem when $r=0$ is known, for the problem then reduces to that of the symmetrical spheres already referred to. In the present paper we shall be content to assume that r is small and to find the solution as far as r^2 only, neglecting r^4, r^6, etc.

In addition to the rectangular axes x, y, z, fixed in space, let us suppose there to be a system of rectangular axes ξ, η, ζ fixed in each molecule, having the centre of gravity for origin, and such that the coordinates of the geometrical centre are 0, 0, r. The mass of each molecule will be taken to be m, the radius a, and the moment of inertia about the axes ξ, η, ζ will be mk^2, mk^2, mk'^2 respectively.

* "The distribution of molecular energy," *Phil. Trans.* 196, p. 297.

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§ 3. Let us suppose that for any single sphere the directions of the axes ξ, η, ζ are connected with those of the axes x, y, z by the following scheme of direction cosines:—

	ξ	η	ζ
x	l_1,	l_2,	l_3
y	m_1,	m_2,	m_3
z	n_1,	n_2,	n_3

.....(1).

The condition of the sphere at any instant may be regarded as determined by u, v, w, the velocities of the centre of gravity parallel to x, y, z; $\varpi_1, \varpi_2, \varpi_3$, the angular velocities of rotation about the axes ξ, η, ζ, and the nine direction-cosines of scheme (1) of which three only are independent.

There are therefore nine independent variables necessary to determine the condition of a sphere. The kinetic energy is given by

$$2T = m\left[u^2 + v^2 + w^2 + k^2(\varpi_1^2 + \varpi_2^2) + k'^2\varpi_3^2\right] \text{.....(2).}$$

In discussing a collision between two spheres, it is necessary to know not only the conditions of the two spheres, but also λ, μ, ν, the direction-cosines of the line of centres at impact referred to the fixed axes x, y, z. This introduces two more variables, so that a collision requires twenty independent variables for its complete specification.

§ 4. Let us examine first the changes in the nine variables of a sphere, which are produced by an impulse I acting along the radius of which the direction cosines are λ, μ, ν.

Referred to ξ, η, ζ axes, the point of application of this impulse may be taken to be $0, 0, r$. The components of the impulse will be

$$(l_1\lambda + m_1\mu + n_1\nu)\,I, \text{ etc. } \text{.....(3)},$$

and the components of the resulting couple

$$-r(l_2\lambda + m_2\mu + n_2\nu)\,I, \quad r(l_1\lambda + m_1\mu + n_1\nu)\,I, 0 \text{...(4).}$$

Hence, if variables after the impulse are distinguished by a horizontal bar, the new velocities of translation will be given by

$$\bar{u} = u + \frac{I\lambda}{m}, \text{ etc. } \text{.....(5)},$$

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and those of rotation by

$$\left.\begin{aligned}\bar{\varpi}_1 &= \varpi_1 - \frac{r(l_2\lambda + m_2\mu + n_2\nu)}{mk^2} I \\ \bar{\varpi}_2 &= \varpi_2 + \frac{r(l_1\lambda + m_1\mu + n_1\nu)}{mk^2} I \\ \bar{\varpi}_3 &= \varpi_3 \quad \ldots\ldots\ldots\ldots\ldots\end{aligned}\right\} \ldots\ldots\ldots(6).$$

§ 4. Let us next regard this impulse as arising from a second sphere, of which the condition is determined by accented variables u', v', w', ..., the line of centres at impact having direction cosines λ, μ, ν.

Before impact the components parallel to x, y, z of the velocity of the first sphere are

$$u + r(l_1\varpi_2 - l_2\varpi_1),$$
$$v + r(m_1\varpi_2 - m_2\varpi_1),$$
$$w + r(n_1\varpi_2 - n_2\varpi_1).$$

Hence if α, β, γ denote the components of the velocity of the centre of the second sphere, relatively to that of the first, we have

$$\alpha = u' - u + r(l_1'\varpi_2' - l_2'\varpi_1' - l_1\varpi_2 + l_2\varpi_1), \text{ etc.} \quad \ldots(7).$$

The components of relative velocity after impact can be deduced by writing $\bar{u}$, $\bar{u}'$, ... in place of u, u', ... The quantities $\bar{u}$, ... are given by equations (5), (6); the quantities $\bar{u}'$, ... are given by similar equations, except that I must be replaced by $-I$, since the impulse on the second sphere in the direction λ, μ, ν is $-I$. Substituting this value, and comparing with equation (7) we obtain

$$\bar{\alpha} = \alpha - \frac{2I\lambda}{m} - \frac{r^2 I}{mk^2}\{l_1(l_1\lambda + m_1\mu + n_1\nu) + l_2(l_2\lambda + m_2\mu + n_2\nu)$$
$$+ l_1'(l_1'\lambda + m_1'\mu + n_1'\nu) + l_2'(l_2'\lambda + m_2'\mu + n_2'\nu)\} \ldots\ldots(8),$$

and there are similar equations for $\bar{\beta}$, $\bar{\gamma}$.

Multiplying these three equations by λ, μ, ν and adding, we obtain

$$\lambda\bar{\alpha} + \mu\bar{\beta} + \nu\bar{\gamma} = \lambda\alpha + \mu\beta + \nu\gamma - \frac{2I}{m}(1 + Ar^2) \quad \ldots(9),$$

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where

$$A = \frac{1}{2k^2}\{(l_1\lambda + m_1\mu + n_1\nu)^2 + (l_2\lambda + m_2\mu + n_2\nu)^2$$

$$+ (l_1'\lambda + m_1'\mu + n_1'\nu)^2 + (l_2'\lambda + m_2'\mu + n_2'\nu)^2\} \ldots\ldots (10).$$

At the moment of greatest compression, the components of relative velocity will be $\frac{1}{2}(\alpha + \bar{\alpha})$, $\frac{1}{2}(\beta + \bar{\beta})$, $\frac{1}{2}(\gamma + \bar{\gamma})$, and therefore the relative velocity along the line of centres will be

$$\tfrac{1}{2}(\lambda\alpha + \mu\beta + \nu\gamma + \lambda\bar{\alpha} + \mu\bar{\beta} + \nu\bar{\gamma}).$$

This vanishes, and therefore by equation (9)

$$\lambda\alpha + \mu\beta + \nu\gamma = -(\lambda\bar{\alpha} + \mu\bar{\beta} + \nu\bar{\gamma}) = (1 + Ar^2)\frac{I}{m}.$$

Hence, since we are neglecting r^4,

$$\frac{I}{m} = (1 - Ar^2)(\lambda\alpha + \mu\beta + \nu\gamma) \ldots\ldots\ldots\ldots (11).$$

Substituting this value for I in equations (5) and (6), we obtain the velocities of the first sphere after impact.

§ 5. Let us now pass to the consideration of impacts in which the two colliding spheres are in given conditions, but the direction cosines λ, μ, ν may have any values which are consistent with a collision.

The path described by the centre of the second molecule before collision relatively to that of the first is, since the spheres are loaded, not a straight line; but since the relative velocity α, β, γ is the same for all the impacts now being considered, this path is always the same as regards shape and direction.

Imagine the centre of the first sphere reduced to rest and surrounded by a sphere S of radius $2a$. Let the centre of the second sphere describe its path relatively to the first sphere, then whenever an impact occurs this centre must lie on the sphere S.

Let us start from any point P on S and trace the relative path backwards. This path, being a curved line, either may or may not intersect the sphere again in some point R.

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In the former case an impact at P will only be possible if an impact has occurred at some point in PR. In the latter case there is no limitation of this kind, so that the probability of an impact occurring on any small region surrounding P is simply proportional to the projection of this area on a plane perpendicular to the direction of the relative velocity at impact.

Owing to the smallness of r, the relative path will, in general, be only slightly curved. If the maximum curvature of this path is less than that of the sphere S, then obviously so long as the angle between the normal at P and the relative velocity is acute, the relative path will not intersect the sphere S except at P.

§ 6. For the present we shall assume the velocities of the two spheres to be such that this latter condition is satisfied. In this case the proportion out of all collisions, of those for which the angle between the directions of the relative velocity and line of centres lies between ϕ and $\phi + d\phi$, is easily seen to be $2 \sin\phi \cos\phi \, d\phi$, the limits for ϕ being 0 and $\frac{1}{2}\pi$.

By $[q]$ we shall denote the average value of q taken over all collisions in which the two spheres are, before impact, in certain specified conditions.

From equation (5)

$$\bar{u}^2 + \bar{v}^2 + \bar{w}^2 = u^2 + v^2 + w^2 + \frac{2I}{m}(\lambda u + \mu v + \nu w) + \frac{I^2}{m^2}, \quad \ldots(12).$$

Before averaging this equation, we must calculate

$$\left[\frac{I^2}{m^2}\right] \text{ and } \left[\frac{I}{m}(\lambda u + \mu v + \nu w)\right].$$

If V is the relative velocity

$$\lambda\alpha + \mu\beta + \nu\gamma = V\cos\phi \quad \ldots\ldots\ldots\ldots(13);$$

hence by equation (11), we have as far as r^3,

$$\frac{I^2}{m^2} = V^2\cos^2\phi - 2r^3 A V^2 \cos^3\phi.$$

Now

$$[V^2\cos^2\phi] = V^2 \int_0^{\frac{1}{2}\pi} 2\sin\phi\cos^2\phi \, d\phi = \tfrac{1}{2}V^2.$$

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Hence

$$\left[\frac{I^2}{m^2}\right] = \tfrac{1}{2}V^2 - 2r^2\left[AV^2\cos^2\phi\right] \quad \ldots\ldots\ldots\ldots (14).$$

Again, from equation (11),

$$\left[\frac{I}{m}(\lambda u + \mu v + \nu w)\right] = \Sigma\left[\lambda^2\right]\alpha u + \Sigma\left[\mu\nu\right](\beta w + \gamma v)$$

$$- r^2\left\{\Sigma\left[A\lambda^2\right]\alpha u + \Sigma\left[A\mu\nu\right](\beta w + \gamma v)\right\} \ldots\ldots (15).$$

§ 7. If we take the direction of the relative velocity for pole, and denote the coordinates of a point on a unit sphere, referred to this pole, by Θ, Φ, then the probability that the direction (λ, μ, ν) meets this unit sphere within limits $d\Theta\, d\Phi$ is $\frac{1}{\pi}\sin\Theta\cos\Theta\, d\Theta\, d\Phi$. Hence

$$\left[\lambda^2\right] = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\lambda^2\sin\Theta\cos\Theta\, d\Theta\, d\Phi \ldots\ldots (16).$$

If we take the coordinates of the axes of x to be $(\theta, 0)$, we have

$$\lambda = \cos\Theta\cos\theta - \sin\Theta\sin\theta\cos\Phi \quad \ldots\ldots\ldots (17).$$

Substituting this value for λ in equation (16) and integrating, we obtain

$$\left[\lambda^2\right] = \tfrac{1}{2}\cos^2\theta + \tfrac{1}{4}\sin^2\theta$$

$$= \tfrac{1}{4}(1 + \cos^2\theta)$$

$$= \tfrac{1}{4}\left(1 + \frac{\alpha^2}{V^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (18),$$

and similar equations give $[\mu^2]$, $[\nu^2]$.

If we suppose the axis of y to be (θ', ϕ'), then

$$\mu = \cos\Theta\cos\theta' - \sin\Theta\sin\theta'\cos(\Phi - \phi'),$$

and multiplying this by the respective sides of equation (17) and integrating, we obtain

$$\left[\lambda\mu\right] = \tfrac{1}{4}\cos\theta\cos\theta' = \frac{\alpha\beta}{4V^2} \quad \ldots\ldots\ldots\ldots (19),$$

and similar equations give $[\mu\nu]$, $[\nu\lambda]$.

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§ 8. Substituting these values in equation (15),

$$\left[\frac{I}{m}(\lambda u+\mu v+\nu w)\right]=\tfrac{1}{4}\Sigma\alpha u+\tfrac{1}{4}\Sigma\frac{\alpha^2 u}{V^2}+\tfrac{1}{4}\Sigma\frac{\beta\gamma(\beta w+\gamma v)}{V^2}-r^3 D,$$

where

$$D=\Sigma[A\lambda^2]\alpha u+\Sigma[A\mu\nu](\beta w+\gamma v) \quad \ldots\ldots(20).$$

Or, simplifying,

$$\left[\frac{I}{m}(\lambda u+\mu v+\nu w)\right]=\tfrac{1}{2}(\alpha u+\beta v+\gamma w)-r^3 D,$$

and hence from equation (12),

$$[\bar{u}^2+\bar{v}^2+\bar{w}^2]=u^2+v^2+w^2+\alpha u+\beta v+\gamma w+\tfrac{1}{2}V^2 -2r^3(D+[AV^2\cos^2\phi]).....(21).$$

On the right-hand side the upper line

$$=\tfrac{1}{2}\{u^2+v^2+w^2+(u+\alpha)^2+(v+\beta)^2+(w+\gamma)^2\}$$
$$=\tfrac{1}{2}\Sigma u^2+\tfrac{1}{2}\Sigma\{u'+r(l_1'\varpi_2'-l_2'\varpi_1'-l_1\varpi_2+l_2\varpi_1)\}^2$$

by equation (7),

$$=\tfrac{1}{2}\Sigma(u^2+u'^2)+r\Sigma u'(l_1'\varpi_2'-l_2'\varpi_1'-l_1\varpi_2+l_2\varpi_1) +\tfrac{1}{2}r^2(\varpi_1^2+\varpi_2^2+\varpi_1'^2+\varpi_2'^2) -r^2\Sigma(l_1l_2\varpi_1\varpi_2-l_1l_2'\varpi_1'\varpi_2+\ldots).$$

If therefore we write

$$u^2+v^2+w^2=c^2,$$
$$\bar{u}^2+\bar{v}^2+\bar{w}^2=\bar{c}^2,$$
$$\varpi_1^2+\varpi_2^2=\varpi^2,$$
$$\varpi_1^2+\bar{\varpi}_2^2=\bar{\varpi}^2,$$

and adopt a similar notation in accented symbols, equation (21) may be written

$$[\bar{c}^2]=\tfrac{1}{2}(c^2+c'^2)+\tfrac{1}{2}r^2(\varpi^2+\varpi'^2)+r\Sigma u'(l_1'\varpi_2'-\ldots) -r^2\Sigma(l_1l_2\varpi_1\varpi_2-\ldots)-2r^3(D+[AV^2\cos^2\phi])\ldots(22).$$

§ 9. We must now carry the process of averaging still further, so as to apply to all possible positions of the *axes* of

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the colliding spheres; that is to say, we must average over all collisions determined by given values of the twelve coordinates

$$u,\ v,\ w,\ u',\ v',\ w',\ \varpi_1,\ \varpi_2,\ \varpi_3,\ \varpi_1',\ \varpi_2',\ \varpi_3' \ldots\ldots (23).$$

Let the result of averaging in this manner be denoted by $\{q\}$.

When we require to average any quantity p over all possible positions of the axes of the first sphere, all positions being regarded as equally probable, we shall denote the result by $\int p d\omega$, and the same quantity averaged over all positions of the axes of the second sphere will be denoted by $\int p d\omega'$. The probability of an impact occurring for any given positions of the axes is, however, proportional to V, and this depends on the positions of the axes. Thus averaging over all collisions for which the coordinates (23) have given values, we shall have

$$\{q\} = \frac{\iint q V d\omega\, d\omega'}{\iint V d\omega\, d\omega'} \ldots\ldots\ldots\ldots\ldots (24).$$

To calculate $\{\dot{c}^2\}$ from equation (22), we must first evaluate $\{l_1\}\ldots$, $\{l_1 l_1'\}\ldots$, $\{D\}$, and $\{A V^2 \cos^2\phi\}$. Since, however, we are neglecting r^2, we require $\{l_1\}\ldots$ only as far as terms in r, and $\{l_1 l_1'\}\ldots$ and the remaining averages only when $r = 0$.

From equation (7)

$$V^2 = \Sigma \{u' - u + r(l_1'\varpi_2' - l_2'\varpi_1' - l_1\varpi_2 + l_2\varpi_1)\}^2,$$

so that if we write

$$U^2 = (u' - u)^2 + (v' - v)^2 + (w' - w)^2$$

we have, as far as terms in r,

$$V = U - \frac{r}{U}\Sigma (u' - u)(l_1'\varpi_2' - l_2'\varpi_1' - l_1\varpi_2 + l_2\varpi_1).$$

Now obviously $\iint p d\omega\, d\omega' = 0$, when p has any of the values $l_1, l_2, \ldots, l_1', l_2', \ldots, l_1 l_1', \ldots$, etc., so that $\iint V d\omega\, d\omega' = U$ as far as terms in r, and hence, from formula (24),

$$\{l_1'\} = \frac{\iint \left(-\frac{r}{U} l_1'^2 \varpi_2' (u' - u)\right) d\omega\, d\omega'}{U},$$

and since obviously $\iint l_1^2 d\omega\, d\omega' = \frac{1}{3}$, this becomes

$$\{l_1'\} = -\frac{r\varpi_2'(u' - u)}{3U^2}.$$

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With the help of this and similar other expressions, we have as the averaged value of one term in equation (22)

$$r\Sigma\{u'(l_1'\varpi_2'-\ldots)\}=-\frac{r^2(\varpi^2+\varpi'^2)}{3U^2}\Sigma u'(u'-u).$$

The remaining averages are required only when $r=0$. We may therefore write U for V, and $u'-u$ for α, etc.

It is obvious at once that when $r=0$, $\{l_1l_1'\}=0$, and hence that as the averaged value of a further term in equation (22) we have

$$\{r^2\Sigma(l_1l_2\varpi_1\varpi_2-\ldots)\}=0.$$

As regards the remaining terms, putting $r=0$,

$$\{D\}=\Sigma\{[A\lambda^2]\alpha u\}+\Sigma\{[A\mu v](\beta w+\gamma v)\}\quad\ldots(25)$$

by equation (20); and since the various systems of quantities may now be supposed to vary independently

$$\{[A\lambda^2]\alpha u\}=\{A\}\{\lambda^2\}(u'-u)u$$

$$=\tfrac{1}{4}\left(1+\frac{U^2}{(u'-u)^2}\right)(u'-u)u\{A\},$$

$$\{[A\mu v](\beta w+\gamma v)\}=\{(v-v')w+(w'-w)v\}\frac{(v'-v)(w'-w)}{4U^2}\{A\},$$

etc., etc., and from equation (10)

$$\{A\}=\frac{1}{2k^2}(\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{3})=\frac{2}{3k^2}.$$

Substituting these values in equation (25), we find that

$$\{D\}=\frac{1}{3k^2}\Sigma u(u'-u),$$

and in a similar way we have

$$\{AV^2\cos^2\phi\}=\{A\}U^2\{\cos^2\phi\}$$

$$=\frac{V^2}{3k^2}.$$

Hence, averaging equation (22), we obtain

$$\{\bar{c}^2\}=\tfrac{1}{2}(c^2+c'^2)+\tfrac{1}{2}r^2(\varpi^2+\varpi'^2)-\frac{r^2(\varpi^2+\varpi'^2)}{3U^2}\Sigma u'(u'-u)$$

$$-\frac{2r^2}{3k^2}(\Sigma u(u'-u)+U^2).$$

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There is an exactly similar equation for $\{\bar{c}^2\}$ and by addition of these two equations we obtain

$$\{\bar{c}^2 + \bar{c}'^2\} = c^2 + c'^2 + \tfrac{4}{3}r^2(\varpi^2 + \varpi'^2) - \frac{2r^2}{3k^2}U^2 \quad \ldots(26).$$

If we use the symbol Δ to denote the increase, caused by collision, of the quantity to which it is applied, we have

$$\{\Delta(c^2 + c'^2)\} = r^2\left\{\tfrac{4}{3}(\varpi^2 + \varpi'^2) - \frac{2V^2}{3k^2}\right\} \quad \ldots\ldots(27).$$

§ 10. So long as we remain in ignorance of the law of distribution of coordinates, it is impossible to carry any further the process of direct averaging. We may, however, proceed as follows.

Imagine a volume Ω in which are distributed a great number N of molecules of the kind we have been considering. An expression has just been found for the mean change of translational energy produced by collisions in which the molecules have given velocities. A summation extending over all collisions which occur during a short interval of time dt will lead to an expression for the total change in the translational energy of the gas during that period.

From the form of equation (27) it is clear that this expression will contain r^2 as a factor. But as we are neglecting powers of r of a degree higher than the second, we may put $r = 0$ in all terms multiplied by r^2. It follows that to find $\Sigma\Delta(c^2 + c'^2)$ as far as r^2, we need only calculate $\Sigma(\varpi^2 + \varpi'^2)$ and ΣU^2 on the assumption that $r = 0$, the summation extending in each case to all the collisions which occur in the time dt.

This neglects a correction which is required by the considerations mentioned in § 5, a correction which would consist in adding to the expression for $\Sigma\Delta(c^2 + c'^2)$, a quantity equal to the sum of all the separate corrections to be applied to the right-hand member of equation (27) in cases in which this equation fails. Now each separate correction will clearly contain r^2 as a factor, and on averaging this must be further multiplied by a factor proportional to the number of cases in which the correction is required; *i.e.* by a factor which vanishes when $r = 0$. The correction to the final result is therefore of a higher order of small quantities than r^2, and may therefore be neglected.

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on for $[c^2]_{\Sigma U^2}$. For a similar reason we may, in calculating $\Sigma(\varpi^2+\varpi'^2)$ assume the law of distribution of velocities to be that which would obtain in the case of $r=0$. We therefore assume the number of molecules for which c and ϖ lie within ranges dc, $d\varpi$ to be

$$4\pi A c^2 e^{-hmc^2} f(\varpi)\, dc\, d\varpi \quad \ldots\ldots\ldots\ldots\ldots\ldots (28).$$

It is not necessary to make any assumptions as to the form of $f(\varpi)$, but we shall suppose that $\int_0^\infty f(\varpi)\, d\varpi = 1$, any constant multiplier being absorbed into the constant A, so that, if N is the total number of molecules,

$$N = A\sqrt{\left(\frac{\pi^3}{h^3 m^3}\right)}.$$

If θ be the angle between the directions of c and c', we have

$$U^2 = c^2 + c'^2 - 2cc'\cos\theta,$$

and, keeping c and c' constant,

$$U\,dU = cc'\sin\theta\, d\theta.$$

The number of collisions occurring in time dt for which c, c', ϖ, ϖ', θ lie within ranges dc, dc', $d\varpi$, $d\varpi'$, $d\theta$ is

$$16\pi^2 A^2 c^2 c'^2 e^{-hm(c^2+c'^2)} f(\varpi) f(\varpi')\, dc\, dc'\, d\varpi\, d\varpi' \frac{\pi a^2 U dt}{\Omega} \frac{\sin\theta\, d\theta}{2}$$

$$= \frac{8\pi^3 a^2 A^2 dt}{\Omega} cc' e^{-hm(c^2+c'^2)} f(\varpi) f(\varpi')\, U^2\, dc\, dc'\, d\varpi\, d\varpi'\, dU.$$

Hence, we have

$$\Sigma(\varpi^2+\varpi'^2) = \frac{8\pi^3 a^2 A^2 dt}{\Omega} \iint (\varpi^2+\varpi'^2) f(\varpi) f(\varpi')\, d\varpi\, d\varpi'$$

$$\times \iint e^{-hm(c^2+c'^2)} cc'\, U^2\, dc\, dc'\, dU \ldots (29),$$

and $$\Sigma U^2 = \frac{8\pi^3 a^2 A^2 dt}{\Omega} \iint e^{-hm(c^2+c'^2)} cc'\, U^4\, dc\, dc'\, dU \ldots (30).$$

If we denote by ψ the mean value of ϖ^2 arranged over all molecules, namely, $\int \varpi^2 f(\varpi)\, d\varpi$, then the first integral in (29) is clearly equal to 2ψ. The second integral in (29) and the

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integral in (30) can be evaluated in the usual way,* their values being respectively

$$\tfrac{1}{4}\sqrt{\left(\frac{\pi}{2h^7m^7}\right)} \text{ and } \sqrt{\left(\frac{\pi}{2h^5m^5}\right)}.$$

Hence we obtain from equation (27)

$$\Sigma\Delta\,(c^2+c'^2)=\frac{8\pi^2a^2A^2dt}{\Omega}\,r^2\tfrac{1}{4}\sqrt{\left(\frac{\pi}{2h^7m^7}\right)}\left\{2\psi-\frac{2}{3k^2}\cdot\frac{4}{hm}\right\} \quad \ldots\ldots\ldots(31).$$

Let us denote the mean translational energy of a molecule by k, and the mean rotational energy ($\frac{1}{2}mk^2\varpi^2$) by H, so that

$$K=\frac{3}{4h}, \quad H=\frac{mk^2\psi}{2},$$

and

$$\Sigma\Delta\,(c^2+c'^2)=\frac{2}{m}\,\Sigma\Delta NK=\frac{2N}{m}\,\frac{dK}{dt}\,dt.$$

Substituting these values in equation (31), and writing ρ for mN/Ω, the density, we have

$$\frac{dK}{dt}=\frac{16r^2a^2\rho}{3mk^4}\sqrt{\left(\frac{2\pi}{3m}K\right)}\left(H-\tfrac{2}{3}K\right) \quad \ldots\ldots\ldots(32).$$

§ 12. This equation gives the rate of change in K, this change being produced by the energy transferred at collisions between molecules. All this energy is transformed into rotational energy, so that the value of H experiences a change equal and opposite to that of K. We are, however, supposing a second cause of change in H. We suppose that the rotational energy is continually dissipated away, and hence there is loss to H equal to twice the average value of the dissipation function averaged over all the molecules. This function may be taken to be ϵH. Hence we obtain as the equation giving the rate of change in H

$$\frac{dH}{dt}=-2\epsilon H-\frac{dK}{dt} \quad \ldots\ldots\ldots\ldots\ldots(33).$$

If, for the sake of brevity, we write

$$\beta=\frac{16r^2a^2\rho}{3mk^2}\sqrt{\left(\frac{2\pi}{3m}\right)},$$

* Cf. Boltzmann, *Vorlesungen über gastheorie*, p. 64.

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equations (32) and (33) become

$$\left.\begin{aligned}\frac{dK}{dt} &= \beta\sqrt{K}\,(H-\tfrac{2}{3}K)\\ \frac{dH}{dt} &= -2\varepsilon H-\beta\sqrt{K}\,(H-\tfrac{2}{3}K)\end{aligned}\right\}\quad\ldots\ldots(34).$$

§ 13. In the special case in which $\varepsilon=0$, we have, from these equations,

$$\frac{d}{dt}(H-\tfrac{2}{3}K)=-\tfrac{5}{3}\beta\sqrt{K}(H-\tfrac{2}{3}K).$$

It is therefore clear that from whatever condition the gas starts, it will ultimately attain to a steady state in which

$$H-\tfrac{2}{3}K=0\quad\ldots\ldots(35).$$

Since H is the energy of two momentoids, and K of three, this state is one of equipartition of energy. Thus, by putting $\varepsilon=0$, our calculations give an illustration of the general dynamical theory of a conservative gas.

§ 14. Our aim is now to obtain from the general equations (34) a similar illustration of the general dynamical theory of a non-conservative gas, given in the preceding paper. Let us write

$$u=2\varepsilon H-\tfrac{2}{3}\beta K^{\frac{3}{2}}\quad\ldots\ldots(36),$$

so that

$$\begin{aligned}\frac{du}{dt} &= 2\varepsilon\frac{dH}{dt}-\beta\sqrt{K}\frac{dK}{dt}\\ &= -\frac{u}{2\varepsilon}(4\varepsilon^2+2\varepsilon\beta\sqrt{K}+\beta^2K)-\tfrac{1}{3}\frac{\beta^3K^{\frac{3}{2}}}{\varepsilon}\quad\ldots\ldots(37),\end{aligned}$$

on substitution from equations (34). The general theory does not lead us to expect to find a definite steady state except for small values of K.

Now as K decreases, the relative importance of the three terms of the bracket $(4\varepsilon^2+2\varepsilon\beta\sqrt{K}+\beta^2K)$ changes, until ultimately the term $4\varepsilon^2$ is of preponderating importance. When these values of K are reached, the last term in equation (37) becomes negligible, being of the same order of importance as the term β^2K of the bracket, so that equation (37) assumes the form

$$\frac{du}{dt}=-2\varepsilon u.$$

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It therefore appears that for these values of K, the gas tends towards a state given by $u = 0$, or

$$\epsilon H = \tfrac{1}{3}\beta K^{\frac{3}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38).$$

In this state $H/K = 0$ in the limit when $K = 0$, so that *the whole energy tends to become translational.*

Thus in the generalised space of the preceding paper, it is clear that the stream lines will tend to form into clusters, and the equations of these clusters will be such that equation (38) is satisfied.

§ 15. The solution of equations (34), in the general case in which K is not small, can be represented graphically. If we eliminate the time between these equations, we have

$$\frac{dH}{dK} = -1 - \frac{2\epsilon H}{\beta\rho\sqrt{K}\,(H - \frac{2}{3}K)}.$$

Thus dH/dK vanishes when

$$\beta\rho\sqrt{K}(H - \tfrac{2}{3}K) + 2\epsilon H = 0 \quad \ldots\ldots\ldots\ldots (39),$$

is infinite when

$$H - \tfrac{2}{3}K = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40),$$

or $K = 0$, and is negative, being equal to -1, when $H = 0$. If, then, we measure H, K along rectangular axes, the quadrant in which H and K are both positive will be divided into three sections by the axes and the two curves given by equations (39) and (40). Starting from the axis of K, the signs of dK/dH in these three sections are respectively negative, positive, and negative. The curves are, therefore, roughly as represented in fig. 1. It will be seen that they tend to converge, for small values of H and K, into a single stream line for which, as the origin is approached, we have

$$\text{Lt.}\ \frac{H}{K} = 0 \text{ and Lt.}\ \frac{dH}{dK} = 0.$$

This, then, is in accordance with the results previously obtained.

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A GEOMETRICAL CORRESPONDENCE IN SPACE.

By G. B. Mathews.

IN Jacobi's *Collected Works*, Vol. VII., pp. 42–68, will be found the notes on quadric surfaces which were first published after his death in Crelle's *Journal*, Vol. LXXIII., accompanied by a commentary by O. Hermes. In the present paper I propose to show that Jacobi's construction is really a case of an analytical point-to-point correspondence in space, which can be expressed in one or other of two simple forms. As one point describes any algebraic surface the corresponding point describes another algebraic locus, the order of which is, according to circumstances, either the same as that of the first, or else the double of it. The transformation is, in this respect, similar to inversion.

For the sake of comparison, the notation used will be, on the whole, the same as that of Jacobi; but it has been found convenient to make a few modifications. As Jacobi's construction is seldom referred to in text-books, I have thought it advisable to discuss the matter in detail; of course this involves some repetition of Jacobi's and Hermes' work, but the analysis here given is, I think, rather simpler and more direct than theirs, and leads incidentally to some interesting algebraic theorems.

The fundamental problem is this: any two fixed triangles ABC, $A'B'C'$ being taken in space, two points P, P' are said to correspond when

$$PA, PB, PC = P'A', P'B', P'C'$$

respectively, and it is required to determine the locus of P' when that of P is assigned. It is clear that to each position of P correspond, in general, two positions of P', and conversely: also that P' is not always real when P is. Again, we may, to simplify our analysis, suppose the triangles ABC, $A'B'C'$ placed in any relative position that is found to be convenient.

Ivory's theorem about corresponding points on confocal quadrics leads at once to Jacobi's first proposition, that when P describes the plane ABC the point P' describes a quadric surface. To prove this we have only to show that ABC, $A'B'C'$ can be so placed that (A, A'), (B, B'), (C, C') are three pairs of corresponding points on two confocal co-planar conics.

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Let a, b, c, Δ be the sides and area of ABC, the corresponding parts of $A'B'C'$ being denoted by accented letters: these are the primary data upon which everything else depends. Assume as the equations of two confocal conics

$$z=0, \quad \frac{x^2}{m^2}+\frac{y^2}{n^2}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$z=0, \quad \frac{x^2}{m'^2}+\frac{y^2}{n'^2}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

with $$m^2-m'^2=n^2-n'^2=p^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Let α, β, γ be the eccentric angles of any three points A, B, C on (1), and let A', B', C' be the points on (2) which have the same eccentric angles. Then, if the triangles ABC, $A'B'C'$ are congruent to the given bases, we must have

$$\left.\begin{aligned} a^2 &= 4\sin^2\frac{\beta-\gamma}{2}\left\{m^2\sin^2\frac{\beta+\gamma}{2}+n^2\cos^2\frac{\beta+\gamma}{2}\right\} \\ b^2 &= 4\sin^2\frac{\gamma-\alpha}{2}\left\{m^2\sin^2\frac{\gamma+\alpha}{2}+n^2\cos^2\frac{\gamma+\alpha}{2}\right\} \\ c^2 &= 4\sin^2\frac{\alpha-\beta}{2}\left\{m^2\sin^2\frac{\alpha+\beta}{2}+n^2\cos^2\frac{\alpha+\beta}{2}\right\} \end{aligned}\right\} \ldots(4),$$

with three similar equations obtained by accenting the italics.

Put, with Jacobi,

$$a^2-a'^2,\ b^2-b'^2,\ c^2-c'^2=u, v, w \quad \ldots\ldots\ldots(5);$$

then from (4) and the other set

$$u=4p^2\sin^2\frac{\beta-\gamma}{2},\ v=4p^2\sin^2\frac{\gamma-\alpha}{2},\ w=4p^2\sin^2\frac{\alpha-\beta}{2} \ldots(6).$$

Again,

$$a'^2m^2-a^2m'^2=4\,(m^2n'^2-m'^2n^2)\sin^2\frac{\beta-\gamma}{2}\cos^2\frac{\beta+\gamma}{2} \ldots(7),$$

with two similar equations, and hence

$$\sqrt{(a'^2m^2-a^2m'^2)}\pm\sqrt{(b'^2m^2-b^2m'^2)}\pm\sqrt{(c'^2m^2-c^2m'^2)}=0\ldots(8),$$

an equation to find m/m', which, on being rationalised, is

$$\Delta'^2m^4-(\Delta^2+\Delta'^2-\mu)\,m^2m'^2+\Delta^2m'^4=0 \quad \ldots\ldots(9),$$

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where

$$16\mu = 2vw + 2wu + 2uv - u^2 - v^2 - w^2 \quad \ldots\ldots\ldots(10).$$

(This μ is 1-16th of Jacobi's μ).

The discriminant of (9) is

$$R = \{(\Delta + \Delta')^2 - \mu\}\{(\Delta - \Delta')^2 - \mu\} \quad \ldots\ldots\ldots(11),$$

and it has been shown by Jacobi that R is positive, so that the roots of the quadratic are real. Moreover, the ratio of n^2 to n'^2 is determined by the same equation, as it ought to be, since

$$\Delta : \Delta' = mn : m'n'.$$

We have

$$\Delta^2 = 4m^2n^2 \sin^2\frac{\beta-\gamma}{2}\sin^2\frac{\gamma-\alpha}{2}\sin^2\frac{\alpha-\beta}{2}$$

$$= \frac{uvwm^2n^2}{16p^6} \quad \ldots\ldots\ldots(12)$$

by (6).

Now, by (9),

$$\mu m^2m'^2 = -(m^2 - m'^2)(\Delta'^2m^2 - \Delta^2m'^2)$$

$$= -(m^2 - m'^2)\left(m^2 - \frac{m^2n^2}{n'^2}\right)\Delta'^2$$

$$= \frac{p^4\Delta'^2m^2}{n'^2};$$

hence $\quad p^4\Delta'^2 = \mu m'^2n'^2$

and similarly $\quad p^4\Delta^2 = \mu m^2n^2$ $\quad \ldots\ldots\ldots(13).$

Therefore, by (12) and (13),

$$16\mu p^2 = uvw,$$

$$m^2n^2 = \frac{u^2v^2w^2\Delta^2}{256\mu^3}, \qquad m'^2n'^2 = \frac{u^2v^2w^2\Delta'^2}{256\mu^3}.$$

Finally, $\quad m^2 + n^2 = \dfrac{m^2n^2 - m'^2n'^2 + p^4}{p^4}$

$$= \frac{16\mu}{uvw}\cdot\frac{u^2v^2w^2}{256\mu^3}(\Delta^2 - \Delta'^2 + \mu)$$

$$= \frac{uvw}{16\mu^2}(\Delta^2 - \Delta'^2 + \mu).$$

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Thus m^2, n^2 are the roots of the equation

$$256\mu^2x^2 - 16\mu uvw(\Delta^2 - \Delta'^2 + \mu)x + u^2v^2w^2\Delta'^2 = 0 \ldots (14),$$

while m'^2, n'^2 are the roots of

$$256\mu^2x^2 - 16\mu uvw(\Delta^2 - \Delta'^2 - \mu)x + u^2v^2w^2\Delta'^2 = 0 \ldots (15).$$

From either of these equations, we get

$$\left.\begin{aligned} 256\mu^4 f^4 &= u^2v^2w^2R \\ \text{where} \qquad f^2 &= m^2 - n^2 = m'^2 - n'^2 \end{aligned}\right\} \ldots\ldots\ldots\ldots (16).$$

The required conics, then, are completely determined, except in the case when $\mu = 0$, which has to be discussed separately. If $\mu > 0$ the conics are both ellipses, while if $\mu < 0$ they are both hyperbolas. Jacobi has proved that they are never circles except when ABC, $A'B'C'$ are similar.

We have now to determine the angles α, β, γ. The simplest way is to use equations (4) and the associated set, together with (6). Thus

$$\left.\begin{aligned} \sin^2\frac{\beta-\gamma}{2} &= \frac{u}{4p^2} = \frac{4\mu}{vw} \\ \tan^2\frac{\beta+\gamma}{2} &= \frac{n'^2a^2 - n^2a'^2}{m^2a'^2 - m'^2a^2} = \frac{n^2u - a^2p^2}{m^2u + a^2p^2} \end{aligned}\right\} \ldots (17).$$

Or we may adopt Jacobi's geometrical construction: thus if O is the common centre of the conics S, S', and OD_1, OD_2, OD_3 are the radii of S drawn parallel to BC, CA, AB respectively,

$$OD_1^2 = m^2\sin^2\frac{\beta+\gamma}{2} + n^2\cos^2\frac{\beta+\gamma}{2} = \frac{vwa^2}{16\mu} \ldots (18),$$

and the lengths of the radii OD_1', OD_2', OD_3' similarly drawn for the other conic are given by

$$OD_1'^2 = \frac{vwa'^2}{16\mu} \ldots\ldots\ldots\ldots (19)$$

with two similar equations.

Hence also

$$OD_1^2 - OD_1'^2 = \frac{uvw}{16\mu} = p^2 = OD_2^2 - OD_2'^2 = OD_3^2 - OD_3'^2 \ldots (20),$$

and in the case when μ is positive the points on the auxiliary circle of S which correspond to A, B, C form a triangle of which the sides are proportional to $\sqrt{u}$, $\sqrt{v}$, $\sqrt{w}$.

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We shall now regard the conics S, S' as being *given*, and consider a transformation from which Jacobi's theorems follow as particular cases. Let (x, y, z), (x', y', z') be the coordinates of two points P, P' in space, and let $(\xi, \eta, 0)$, $(\xi', \eta', 0)$ be the coordinates of any two corresponding points A, A' on S and S' respectively.

The fact that PA, $P'A'$ are equal is expressed by

$$(x'-\xi')^2+(y'-\eta')^2+z'^2=(x-\xi)^2+(y-\eta)^2+z^2 \ldots(21),$$

and in virtue of the relations

$$m\xi'=m'\xi, \quad n\eta'=n'\eta, \quad \frac{\xi^2}{m^2}+\frac{\eta^2}{n^2}=\frac{\xi'^2}{m'^2}+\frac{\eta'^2}{n'^2}=1,$$

this is satisfied by putting

$$\left.\begin{aligned} m'x'=mx, \quad n'y'=ny \\ \frac{z^2-z'^2}{m^2-m'^2}=\frac{x^2}{m'^2}+\frac{y^2}{n'^2}-1=\frac{x'^2}{m^2}+\frac{y'^2}{n^2}-1 \end{aligned}\right\} \ldots\ldots(22),$$

equations which, though redundant, are consistent, and in which the coordinates of A, A' do not appear. It is supposed, of course, that $m^2-n^2=m'^2-n'^2$: this is the only condition that the constants have to satisfy. Accordingly these relations are the analytical expression of a correspondence in space, and it is easily seen that its properties agree with those stated at the beginning of this paper.

Suppose, in particular, that $z=0$; then P' describes the surface

$$\frac{x'^2}{m^2}+\frac{y'^2}{n^2}+\frac{z'^2}{m^2-m'^2}=1 \ldots\ldots\ldots\ldots\ldots(E),$$

of which S is a principal section and S' a focal conic: this is Jacobi's first theorem.

Again, if P describes the plane-pair

$$Ax+By\pm Cz+D=0 \ldots\ldots\ldots\ldots\ldots(23),$$

P' describes the locus

$$\left(\frac{Am'x}{m}+\frac{Bn'y}{n}+D\right)^2 - C^2\left\{z^2+(m^2-m'^2)\left(\frac{x^2}{m^2}+\frac{y^2}{n^2}-1\right)\right\}=0 \ldots(24).$$

On the whole, it seems best to consider (24) as corresponding to *each* of the planes (23); thus, if P describes both the planes, there are two points P', in general, corresponding to each

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position of P, and these describe (24) twice over. We might, of course, make the convention that to a real positive z a positive z' (if real) should correspond; but this seems to lead to unnecessary complication.

It will be noticed that the quadric corresponding to any plane touches (E) along a plane section; also that the distances of P' from points on the focal conic of E are equal, each to each, to the distances of P from the corresponding points of the principal section of E.

The size and shape of the surface represented by (24) depend upon the position of the plane (23) with reference to E; or, from Jacobi's point of view, upon the position of the plane with reference to ABC, as well as on the constants of the two given triangles. Jacobi obtains some very elegant results by placing the given triangles not in the same plane, but in another properly chosen relative position. But, before going into this, I propose to consider the normal form of the equations of transformation in the case when $\mu = 0$.

Led by analogy, we start by taking two confocal parabolas

$$\left.\begin{aligned} z &= 0,\ y^2 = 4m(x+m) \\ z &= 0,\ y^2 = 4m'(x+m') \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (25),$$

and consider the points A, A' where they are cut by

$$z = 0, \quad y^2 = 4\lambda_1(\lambda_1 - x) \quad \ldots\ldots\ldots\ldots (26),$$

the quantities m, m', λ_1, to fix the ideas, being taken positive.

For the coordinates of A we find

$$x = \lambda_1 - m, \quad y = 2\sqrt{(\lambda_1 m)},$$

with corresponding values for A'. Suppose now that two other such pairs of points (B, B'), (C, C') are found by changing λ_1 into λ_2, λ_3 respectively; then expressing that ABC, $A'B'C'$ are congruent to two given triangles, we have

$$\left.\begin{aligned} a^2 &= (\lambda_2 - \lambda_3)^2 + 4m\{\lambda_2 + \lambda_3 - 2\sqrt{(\lambda_2\lambda_3)}\} \\ a'^2 &= (\lambda_2 - \lambda_3)^2 + 4m'\{\lambda_2 + \lambda_3 - 2\sqrt{(\lambda_2\lambda_3)}\} \end{aligned}\right\} \quad \ldots\ldots (27),$$

with two other similar pairs of equations.

Keeping our previous notation as far as possible, we have

$$\left.\begin{aligned} u, v, w &= 4(m - m')(\sqrt{\lambda_2} - \sqrt{\lambda_3})^2, \\ 4(m - m')(\sqrt{\lambda_3} - \sqrt{\lambda_1})^2,\ & 4(m - m')(\sqrt{\lambda_1} - \sqrt{\lambda_2})^2 \end{aligned}\right\} \ldots (28),$$

so that, with a proper choice of signs,

$$\sqrt{u} \pm \sqrt{v} \pm \sqrt{w} = 0,$$

and hence

$$\mu = 0,$$

as it ought to be.

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Now

$$2\Delta = \begin{vmatrix} \lambda_1 - m, & 2\sqrt{(\lambda_1 m)}, & 1 \\ \lambda_2 - m, & 2\sqrt{(\lambda_2 m)}, & 1 \\ \lambda_3 - m, & 2\sqrt{(\lambda_3 m)}, & 1 \end{vmatrix},$$

and hence

$$\Delta^2 = (\sqrt{\lambda_2} - \lambda_3)^2 (\sqrt{\lambda_3} - \lambda_1)^2 (\sqrt{\lambda_1} - \sqrt{\lambda_2})^2 m = \frac{muvw}{64 (m - m')^3},$$

and similarly

$$\Delta'^2 = \frac{m' uvw}{64 (m - m')^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29).$$

Hence

$$64 (m - m')^3 = \frac{uvw}{\Delta^2 - \Delta'^2},$$

$$m = \frac{\Delta^2 \sqrt{(uvw)}}{8 (\Delta^2 - \Delta'^2)^{\frac{3}{2}}}, \quad m' = \frac{\Delta'^2 \sqrt{(uvw)}}{8 (\Delta^2 - \Delta'^2)^{\frac{3}{2}}} \ldots\ldots\ldots (30),$$

equations which determine the parabolas; while from (27) we obtain

$$\left.\begin{aligned} (\lambda_2 - \lambda_3)^2 &= \frac{a'^2 \Delta^2 - a^2 \Delta'^2}{\Delta^2 - \Delta'^2} \\ (\lambda_3 - \lambda_1)^2 &= \frac{b'^2 \Delta^2 - b^2 \Delta'^2}{\Delta^2 - \Delta'^2} \\ (\lambda_1 - \lambda_2)^2 &= \frac{c'^2 \Delta^2 - c^2 \Delta'^2}{\Delta^2 - \Delta'^2} \end{aligned}\right\} \ldots\ldots\ldots\ldots (31),$$

from which λ_1, λ_2, λ_3 can be found.

If now we write

$$(x - \lambda + m)^2 + \{y - 2\sqrt{(\lambda m)}\}^2 + z^2$$
$$= (x' - \lambda + m')^2 + \{y' - 2\sqrt{(\lambda m')}\}^2 + z'^2,$$

this is satisfied for all values of λ by putting

$$\left.\begin{aligned} x - m &= x' - m', \quad y\sqrt{m} = y'\sqrt{m'} \\ z'^2 - z^2 &= 4 (m - m') (x + m') - \frac{m - m'}{m'} y^2 \\ &= 4 (m - m') (x' + m) - \frac{m - m'}{m} y'^2 \end{aligned}\right\} \ldots (32),$$

and this is the normal form of the transformation when $\mu = 0$.

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If P describes the plane $z=0$, P' will describe the paraboloid

$$\frac{y'^2}{m} + \frac{z'^2}{m-m'} = 4(x'+m) \quad \ldots\ldots\ldots\ldots\ldots\ldots (G),$$

and the transformation may be taken as defined by this surface, just as the other transformation may be taken as defined by (E). The constants m, m' are now independent.

Let us now return to the case when μ does not vanish, and when P describes a plane different from that of ABC. It is evident that the locus of P' is symmetrical with respect to the plane of $A'B'C'$; hence there will be a gain of simplicity if we put $A'B'C'$ in the plane which P describes and take this as the plane $z=0$. By shifting $A'B'C'$ about we may expect to be able to make the quadric locus go through A, B, C, and then a consideration of the first and simplest case suggests that (A, A'), (B, B'), (C, C') will be pairs of corresponding points on the quadric and the degenerate surface in $z=0$ which is bounded by the focal conic in that plane. Hence, we start by assuming

$$\frac{x^2}{m^2} + \frac{y^2}{n^2} + \frac{z^2}{p^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (33),$$

$$\frac{x^2}{m'^2} + \frac{y^2}{n'^2} = 1, \ z=0 \quad \ldots\ldots\ldots\ldots (34),$$

with $m^2 - m'^2 = n^2 - n'^2 = p^2$, and try to determine three sets of direction-cosines $(\alpha, \beta, \gamma)_{1,2,3}$ such that, taking the coordinates of A to be $(m\alpha_1, n\beta_1, p\gamma_1)$, and those of A' to be $(m'\alpha_1, n'\beta_1)$, with corresponding values for the other pairs, we may have ABC, $A'B'C'$ congruent to the given triangles.

The distances of A, B, C from $z=0$ are given; let them be z_1, z_2, z_3; then the conditions to be satisfied are

$$\left.\begin{aligned} p\gamma_1 = z_1, \quad p\gamma_2 = z_2, \quad p\gamma_3 = z_3 & \\ m^2(\alpha_2-\alpha_3)^2 + n^2(\beta_2-\beta_3)^2 + p^2(\gamma_2-\gamma_3)^2 &= a^2 \\ m'^2(\alpha_2-\alpha_3)^2 + n'^2(\beta_2-\beta_3)^2 &= a'^2 \end{aligned}\right\} \ldots (35),$$

with two other pairs which need not be writtten down.

We find

$$u = a^2 - a'^2 = 2p^2(1 - \alpha_2\alpha_3 - \beta_2\beta_3 - \gamma_2\gamma_3),$$

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whence

$$\left.\begin{aligned} \alpha_3\alpha_4 + \beta_3\beta_4 + \gamma_3\gamma_4 &= 1 - \frac{u}{2p^2} \\ \text{and similarly} \quad \alpha_2\alpha_4 + \beta_2\beta_4 + \gamma_2\gamma_4 &= 1 - \frac{v}{2p^2} \\ \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 &= 1 - \frac{w}{2p^2} \end{aligned}\right\} \quad \ldots\ldots\ldots(36).$$

Thus the six quantities

$$\frac{z_1}{p},\ \frac{z_2}{p},\ \frac{z_3}{p},\ 1-\frac{u}{2p^2},\ 1-\frac{v}{2p^2},\ 1-\frac{w}{2p^2}$$

are the cosines of the six arcs joining four points on a sphere; and applying the identity which connects them (Salmon, *Solid Geometry*, p. 34) we get an equation for p, namely,

$$\begin{vmatrix} 1, & \frac{z_1}{p}, & \frac{z_2}{p}, & \frac{z_3}{p} \\ \frac{z_1}{p}, & 1, & 1-\frac{w}{2p^2}, & 1-\frac{v}{2p^2} \\ \frac{z_2}{p}, & 1-\frac{w}{2p^2}, & 1, & 1-\frac{u}{2p^2} \\ \frac{z_3}{p}, & 1-\frac{v}{2p^2}, & 1-\frac{u}{2p^2}, & 1 \end{vmatrix} = 0,$$

which reduces to

$$\begin{aligned} 4p^2\{4\mu - uz_1^2 - vz_2^2 - wz_3^2 + (v+w-u)\,z_2z_3 \\ + (w+u-v)\,z_3z_1 + (u+v-w)\,z_1z_2\} \\ = (uvw - u^2z_1^2 - v^2z_2^2 - w^2z_3^2 + 2vwz_2z_3 + 2wuz_3z_1 + 2uvz_1z_2) \ldots(37). \end{aligned}$$

With Jacobi, let us introduce the triangle $A_1B_1C_1$ which is the projection of ABC upon $z=0$; and let a_1, Δ_1, etc., denote for it the values corresponding to a, Δ, etc., for ABC. Further, let

$$\left.\begin{aligned} &u_1,\ v_1,\ w_1 = a_1^2 - a'^2,\ b_1^2 - b'^2,\ c_1^2 - c'^2 \\ \text{and} \quad &16\mu_1 = 2v_1w_1 + 2w_1u_1 + 2u_1v_1 - u_1^2 - v_1^2 - w_1^2 \end{aligned}\right\} \ldots(38).$$

Then it is found without difficulty that

$$\begin{aligned} 16\mu_1 p^2 = u_1v_1w_1 + 2v_1w_1z_2z_3 + 2w_1u_1z_3z_1 + 2u_1v_1z_1z_2 \\ - u_1^2z_1^2 - v_1^2z_2^2 - w_1^2z_3^2 \quad \ldots\ldots\ldots(39), \end{aligned}$$

in agreement with Jacobi's result.

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Having obtained p', the required direction-cosines are found either by calculation or by a construction on a sphere, making use of (35) and (36). It does not seem worth while to work out their analytical values.

To find m^2, n^2, we may proceed thus. We have

$$4\Delta'^2 = m'^2 n'^2 \begin{vmatrix} \alpha_1, & \beta_1, & 1 \\ \alpha_2, & \beta_2, & 1 \\ \alpha_3, & \beta_3, & 1 \end{vmatrix}^2$$

$$= m'^2 n'^2 \begin{vmatrix} 2 - \frac{z_1^2}{p^2}, & 2 - \frac{w + 2z_1 z_2}{2p^2}, & 2 - \frac{v + 2z_3 z_1}{2p^2} \\ 2 - \frac{w + 2z_1 z_2}{2p^2}, & 2 - \frac{z_2^2}{p^2}, & 2 - \frac{u + 2z_2 z_3}{2p^2} \\ 2 - \frac{v + 2z_3 z_1}{2p^2}, & 2 - \frac{u + 2z_1 z_2}{2p^2}, & 2 - \frac{z_3^2}{p^2} \end{vmatrix}$$

$$= m'^2 n'^2 K,$$

where

$$K = \frac{8\mu - 2(uz_1^2 + vz_2^2 + wz_3^2) + 2(v + w - u) z_2 z_3 + \ldots + \ldots}{p^4} - \frac{uvw + 2(vwz_2 z_3 + \ldots) - (u^2 z_1^2 + \ldots)}{4p^6}$$

$$= \frac{4\mu_1}{p^4},$$

on reduction. Hence

$$m'^2 n'^2 = \frac{\Delta'^2 p^4}{\mu_1},$$

and

$$m^2 n^2 = \frac{\Delta_1^2 p^4}{\mu_1}$$

by similar reasoning. Consequently m^2, n^2 are the roots of the quadratic

$$\mu_1 x^2 - (\Delta_1^2 - \Delta'^2 + \mu_1) p^2 x + \Delta_1^2 p^4 = 0,$$

while m'^2, n'^2 are the roots of

$$\mu_1 x^2 - (\Delta_1^2 - \Delta'^2 - \mu_1) p^2 x + \Delta'^2 p^4 = 0.$$

In comparing these results with Jocobi's, it must be remembered that in this context he writes 16μ, Δ for μ_1, Δ_1.

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ON A QUASI-GEOMETRICAL VIEW OF THE SOLUTION OF A PFAFFIAN EQUATION.

By J. BRILL, M.A.

PART II.*

§ 7. IN the First Part I gave some discussion of the equation in $2m$ variables with its first conditioned case. As a preliminary to a general discussion of conditioned cases it will be necessary to treat in a similar manner the equation in $2m+1$ variables. In this case the normal form of the Pfaffian expression, on the left-hand side of equation (1), is

$$d\beta + l_1 d\alpha_1 + l_2 d\alpha_2 + \ldots + l_m d\alpha_m,$$

where the α's satisfy m equations of the type

$$[234\ldots(2m+1)]\frac{\partial\alpha}{\partial x_1} - [134\ldots(2m+1)]\frac{\partial\alpha}{\partial x_2} + \ldots + (-1)^{2m}[123\ldots(2m)]\frac{\partial\alpha}{\partial x_{2m+1}} = 0 \ldots (15),$$

and $\frac{1}{2}m(m-1)$ equations of the type

$$\Sigma \pm [0123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(2m+1)]\frac{\partial(\alpha, \alpha')}{\partial(x_p, x_q)} = 0 \ldots\ldots\ldots\ldots (16),$$

where it is understood that any particular term under the Σ is affected with a positive or negative sign according as the number of displacements required to restore the numbers

$$p, q, 1, 2, 3, \ldots, p-1, p+1, \ldots, q-1, q+1, \ldots, 2m+1$$

to their natural numerical order is even or odd.

The equation determining β is

$$[234\ldots(2m+1)]\frac{\partial\beta}{\partial x_1} - [134\ldots(2m+1)]\frac{\partial\beta}{\partial x_2} + \ldots + (-1)^{2m}[123\ldots(2m)]\frac{\partial\beta}{\partial x_{2m+1}} = [0123\ldots(2m+1)] \ldots (17).$$

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* For Part I. see *Quarterly Journal of Mathematics*, Vol. XXXIII., p. 257.

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In addition, β is connected with the α's by m equations of the type

$$\Sigma \pm [0123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(2m+1)] \frac{\partial(\beta, \alpha)}{\partial(x_p, x_q)} = 0 \ldots\ldots\ldots\ldots\ldots (18),$$

the same rule obtaining, with regard to the sign of any particular term under the Σ, as in the case of the equations of the type (16).

For the auxiliary system of the equations of the type (15) we have

$$\frac{dx_1}{[234\ldots(2m+1)]} = \frac{-dx_2}{[134\ldots(2m+1)]} = \ldots = \frac{(-1)^m dx_{2m+1}}{[123\ldots(2m)]} \quad \ldots\ldots (19),$$

and for that of equation (17)

$$\frac{dx_1}{[234\ldots(2m+1)]} = \frac{-dx_2}{[134\ldots(2m+1)]} = \ldots = \frac{(-1)^m dx_{2m+1}}{[123\ldots(2m)]} = \frac{d\beta}{[0123\ldots(2m+1)]} \quad \ldots\ldots\ldots\ldots\ldots (20).$$

Equations (19) determine an $(n-1)$-tuply infinite set of curves, which is quite definite. We, however, encounter a striking difference from the state of affairs in the former case, as these curves are not solutions of equation (1). In fact, taking account of the relation

$$X_1[234\ldots(2m+1)] - X_2[134\ldots(2m+1)] + \ldots + (-1)^m X_{2m+1}[123\ldots(2m)] = [0123\ldots(2m+1)],$$

we deduce from equations (20)

$$X_1 dx_1 + X_2 dx_2 + \ldots + X_{2m+1} dx_{2m+1} = d\beta,$$

as is otherwise evident from the normal form, since the α's are constant along the curves of the system.

If we take the symbols

$$\beta_1, \beta_2, \ldots, \beta_{2m}$$

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to denote the $n-1$ independent solutions of equation (15), we have

$$\left.\begin{aligned}
[234\ldots(2m+1)] &= k\frac{\partial(\beta_1, \beta_2, \ldots, \beta_{2m})}{\partial(x_2, x_3, \ldots, x_{2m+1})}\\
[134\ldots(2m+1)] &= k\frac{\partial(\beta_1, \beta_2, \ldots, \beta_{2m})}{\partial(x_1, x_3, x_4, \ldots, x_{2m+1})}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
[123\ldots(2m)] &= k\frac{\partial(\beta_1, \beta_2, \ldots, \beta_{2m})}{\partial(x_1, x_2, \ldots, x_{2m})}
\end{aligned}\right\} \ldots(21);$$

and, combining these with the relation

$$X_1[234\ldots(2m+1)] - X_2[134\ldots(2m+1)] + \ldots$$
$$+ (-1)^{2m} X_{2m+1}[123\ldots(2m)] = [0123\ldots(2m+1)],$$

we have

$$[0123\ldots(2m+1)] = k\frac{\partial(\beta, \beta_1, \beta_2, \ldots, \beta_{2m})}{\partial(x_1, x_2, \ldots, x_{2m+1})}.$$

Further, since

$$\frac{\partial}{\partial x_1}[234\ldots(2m+1)] - \frac{\partial}{\partial x_2}[134\ldots(2m+1)] + \ldots$$
$$+ (-1)^{2m}\frac{\partial}{\partial x_{2m+1}}[123\ldots(2m)] = 0,$$

we obtain from equations (21)

$$\frac{\partial(k, \beta_1, \beta_2, \ldots, \beta_{2m})}{\partial(x_1, x_2, \ldots, x_{2m+1})} = 0.$$

Thus $$k = f(\beta_1, \beta_2, \ldots, \beta_{2m});$$

and, since the β's determine our $(n-1)$-tuply infinite set of curves, as defined by equations (19), we see that k is, in general, the parameter of an $(n-1)$-tuple locus formed by an $(n-2)$-tuply infinite set of the curves of the system. It is clear that the β's may be so chosen that k becomes unity.

Since $[0123\ldots(2m+1)]$ is in general finite, the locus whose parameter is β will not belong to the ensemble of those that

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can be formed from the system of curves defined by equations (19). However, the locus

$$[0123\ldots(2m+1)] = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

will be such that, at any point of it, the particular curve of our system passing through that point will be tangential to the locus of parameter β passing through the same point.

If we suppose u to be the parameter of an $(n-1)$-tuple locus within the n-fold continuum constituted by the system of independent variables, then we obtain, as in Part I.,

$$\frac{d_{12\ldots(2m)}(x_2, x_3, \ldots, x_{2m+1})}{\frac{\partial u}{\partial x_1}} = \frac{d_{12\ldots(2m)}(x_1, x_3, \ldots, x_{2m+1})}{-\frac{\partial u}{\partial x_2}}$$

$$= \frac{d_{12\ldots(2m)}(x_1, x_2, x_4, \ldots, x_{2m+1})}{\frac{\partial u}{\partial x_3}} = \&c.;$$

and, by equations (21), each of these equal ratios is seen to be equal to

$$\frac{\Sigma[234\ldots(2m+1)]\, d_{12\ldots(2m)}(x_2, x_3, \ldots, x_{2m+1})}{k\,\frac{\partial(u, \beta_1, \beta_2, \ldots, \beta_{2m})}{\partial(x_1, x_2, x_3, \ldots, x_{2m+1})}}.$$

Thus, if the locus whose parameter is u be constituted by an $(n-2)$-tuply infinite set of the curves defined by equations (19), we shall have

$$\Sigma[234\ldots(2m+1)]\, d_{12\ldots(2m)}(x_2, x_3, \ldots, x_{2m+1}) = 0.$$

As in Part I., we shall obtain

$$\frac{d_{12\ldots(2m-1)}(x_3, x_4, x_5, \ldots, x_{2m+1})}{\frac{\partial(\alpha, \alpha')}{\partial(x_1, x_2)}} = \frac{d_{12\ldots(2m-1)}(x_2, x_4, x_5, \ldots, x_{2m+1})}{-\frac{\partial(\alpha, \alpha')}{\partial(x_1, x_3)}}$$

$$= \&c.,$$

and thus equation (16) becomes

$$\Sigma[0123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(2m+1)]$$
$$\times d_{12\ldots(2m-1)}(x_1, x_2, \ldots, x_{p-1}, x_{p+1}, \ldots, x_{q-1}, x_{q+1}, \ldots, x_{2m+1})$$
$$= 0 \ldots\ldots (23).$$

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Further, it is clear that equation (18) will lead us to the same result.

It is thus evident that the α's must be so constructed that their elements cause to vanish the last but one, and the last but two of the Multilinear Differential Covariants of the Pfaffian expression forming the left-hand side of equation (1). Further, although the elements of β will satisfy equation (23), they will not, in general, cause the last but one of the covariants to vanish. This will only be the case at such points as fall within the locus defined by equation (22).

If β_0 be a special solution of equation (17), then the general solution is given by the equation

$$\beta = \beta_0 + f(\beta_1, \beta_2, \ldots, \beta_{2m}).$$

Thus if we have two possible values of β, the intersection of the corresponding $(n-1)$-tuple loci will be an $(n-2)$-tuple locus, which may be considered as an $(n-3)$-tuply infinite system of curves selected from the set defined by equations (19).

We must now consider briefly the condition imposed upon u and v, the parameters of two $(n-1)$-tuple loci, in order that the elements of the intersection of the said loci may satisfy equation (23). We have

$$[0345\ldots(2m+1)]$$

$$= \pm \Sigma \left\{ (-1)^{r-1} l_r \frac{\partial(l_1, \ldots, l_{r-1}, l_{r+1}, \ldots, l_m, \alpha_1, \alpha_2, \ldots, \alpha_m)}{\partial(x_3, x_4, x_5, \ldots, x_{2m+1})} \right.$$

$$\left. + \frac{\partial(\beta, l_1, \ldots, l_{r-1}, l_{r+1}, \ldots, l_m, \alpha_1, \ldots, \alpha_{r-1}, \alpha_{r+1}, \ldots, \alpha_m)}{\partial(x_3, x_4, x_5, \ldots, x_{2m+1})} \right\},$$

and so on. Thus the required condition is the vanishing of the expression

$$\Sigma \left\{ (-1)^{r-1} l_r \frac{\partial(l_1, \ldots, l_{r-1}, l_{r+1}, \ldots, l_m, \alpha_1, \alpha_2, \ldots, \alpha_m, u, v)}{\partial(x_1, x_2, \ldots, x_{2m+1})} \right.$$

$$\left. + \frac{\partial(\beta, l_1, \ldots, l_{r-1}, l_{r+1}, \ldots, l_m, \alpha_1, \ldots, \alpha_{r-1}, \alpha_{r+1}, \ldots, \alpha_m, u, v)}{\partial(x_1, x_2, \ldots, x_{2m+1})} \right\}.$$

Since we have

$$[234\ldots(2m+1)] = \pm \frac{\partial(l_1, l_2, \ldots, l_m, \alpha_1, \alpha_2, \ldots, \alpha_m)}{\partial(x_2, x_3, \ldots, x_{2m+1})},$$

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and so on, it is evident that we may take the quantities

$$l_1, l_2, \ldots, l_m, \alpha_1, \alpha_2, \ldots, \alpha_m$$

as a special set of solutions of equation (16). Thus if u and v are solutions of (15) and (16), in addition to satisfying (23), we may write

$$u = \phi(l_1, l_2, \ldots, l_m, \alpha_1, \alpha_2, \ldots, \alpha_m),$$
$$v = \psi(l_1, l_2, \ldots, l_m, \alpha_1, \alpha_2, \ldots, \alpha_m),$$

and our condition reduces to

$$\frac{\partial(\beta, l_1, \ldots, l_m, \alpha_1, \ldots, \alpha_m)}{\partial(x_1, x_2, \ldots, x_{2m+1})} \Sigma \frac{\partial(\phi, \psi)}{\partial(l_r, \alpha_r)} = 0,$$

or, as it may be written,

$$[0123\ldots(2m+1)] \Sigma \frac{\partial(\phi, \psi)}{\partial(l_r, \alpha_r)} = 0.$$

Thus, except in the conditioned cases, we have

$$\Sigma \frac{\partial(\phi, \psi)}{\partial(l_r, \alpha_r)} = 0.$$

§ 8. The next question for discussion is the possibility of obtaining a superficial locus satisfying the system of equations

$$\frac{d_{(2m)(2m+1)}(x_1, x_2)}{[0345\ldots(2m+1)]} = \frac{d_{(2m)(2m+1)}(x_1, x_3)}{-[0245\ldots(2m+1)]} = \&c.$$
$$= \frac{d_{(2m)(2m+1)}(x_p, x_q)}{\pm[0123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(2m+1)]}$$
$$= \&c. \quad \ldots\ldots\ldots\

We have the two identities

$$\begin{aligned}
&[1y_1y_2\cdots y_{2n+1}][23y_1y_2\cdots y_{2n}]\\
&-[2y_1y_2\cdots y_{2n+1}][13y\,y_2\cdots y_{2n}]\\
&+[3y_1y_2\cdots y_{2n+1}][12y_1y_2\cdots y_{2n}]\\
&=[y_1y_2\cdots y_{2n}]\,[123y_1y_2\cdots y_{2n+1}] \quad \ldots\ldots\ldots\ldots \quad (26),
\end{aligned}$$

and

$$\begin{aligned}
&[1y_1y_2\cdots y_{2n+3}][234y_1y_2\cdots y_{2n+1}]\\
&-[2y_1y_2\cdots y_{2n+3}][134y_1y_2\cdots y_{2n+1}]\\
&+[3y_1y_2\cdots y_{2n+3}][124y_1y_2\cdots y_{2n+1}]\\
&-[4y_1y_2\cdots y_{2n+3}][123y_1y_2\cdots y_{2n+1}]\\
&=[y_1y_2\cdots y_{2n+3}][1234y_1y_2\cdots y_{2n+1}y_{2n+3}]\\
&-[y_1y_2\cdots y_{2n+1}y_{2n+3}][1234y_1y_2\cdots y_{2n+3}] \quad \ldots\ldots(27).*
\end{aligned}$$

Now, if in the identity of the type (27) we replace 1, 2, 3, 4 by 2, 3, 4, 5 respectively, and make

$$y_{2n+3}=1, \quad y_{2n+3}=6,$$

and rearrange, we obtain

$$\begin{aligned}
&[126y_1y_2\cdots y_{2n+1}][345y_1y_2\cdots y_{2n+1}]\\
&-[136y_1y_2\cdots y_{2n+1}][245y_1y_2\cdot y_{2n+1}]\\
&+[146y_1y_2\cdots y_{2n+1}][235y_1y_2\cdots y_{2n+1}]\\
&-[156y_1y_2\cdots y_{2n+1}][234y_1y_2\cdots y_{2n+1}]\\
&=-[1y_1y_2\cdots y_{2n+1}][23456y_1y_2\cdots y_{2n+1}]\\
&+[6y_1y_2\cdots y_{2n+1}][12345y_1y_2\cdots y_{2n+1}].
\end{aligned}$$

Further, by interchanging 5 and 6 in this result, we obtain

$$\begin{aligned}
&[125y_1y_2\cdots y_{2n+1}][346y_1y_2\cdots y_{2n+1}]\\
&-[135y_1y_2\cdots y_{2n+1}][246y_1y_2\cdots y_{2n+1}]\\
&+[145y_1y_2\cdots y_{2n+1}][236y_1y_2\cdots y_{2n+1}]\\
&+[156y_1y_2\cdots y_{2n+1}][234y_1y_2\cdots y_{2n+1}]\\
&=[1y_1y_2\cdots y_{2n+1}][23456y_1y_2\cdots y_{2n+1}]\\
&+[5y_1y_2\cdots y_{2n+1}][12346y_1y_2\cdots y_{2n+1}].
\end{aligned}$$

* See "Note on the Algebraic Properties of Pfaffians," *Messenger of Mathematics*, Vol. XXXII, pp. 88–92.

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Adding together these two identities, we have

$$\begin{aligned}
&[125y_1y_2\cdots y_{2m+1}][346y_1y_2\cdots y_{2m+1}]\\
&-[135y_1y_2\cdots y_{2m+1}][246y_1y_2\cdots y_{2m+1}]\\
&+[145y_1y_2\cdots y_{2m+1}][236y_1y_2\cdots y_{2m+1}]\\
&+[126y_1y_2\cdots y_{2m+1}][345y_1y_2\cdots y_{2m+1}]\\
&-[136y_1y_2\cdots y_{2m+1}][245y_1y_2\cdots y_{2m+1}]\\
&+[146y_1y_2\cdots y_{2m+1}][235y_1y_2\cdots y_{2m+1}]\\
&=[5y_1y_2\cdots y_{2m+1}][12346y_1y_2\cdots y_{2m+1}]\\
&+[6y_1y_2\cdots y_{2m+1}][12345y_1y_2\cdots y_{2m+1}] \quad \ldots\ldots\ldots\ (28).
\end{aligned}$$

Now, consider the expression

$$\begin{aligned}
&[01256\ldots(2m+1)][03456\ldots(2m+1)]\\
&-[01356\ldots(2m+1)][02456\ldots(2m+1)]\\
&+[01456\ldots(2m+1)][02356\ldots(2m+1)].
\end{aligned}$$

If we expand the various expressions contained in it, we readily verify that the coefficients of

$$X_1X_2,\ X_1X_3,\ X_1X_4,\ X_2X_3,\ X_2X_4,\ X_3X_4$$

all vanish.

For the coefficient of X_1X_r, where $r>4$, we obtain

$$\begin{aligned}
(-1)^{r-3}\{&[256\ldots(2m+1)][3456\ldots(r-1)\ (r+1)\ldots(2m+1)]\\
&-[356\ldots(2m+1)][2456\ldots(r-1)\ (r+1)\ldots(2m+1)]\\
&+[456\ldots(2m+1)][2356\ldots(r-1)\ (r+1)\ldots(2m+1)]\\
=(-1)^{r-3}&[56\ldots(r-1)\ (r+1)\ldots(2m+1)][234\ldots(2m+1)]
\end{aligned}$$

by means of the identity of the type (26). This result may be obtained by replacing the numbers

$$1,\ 2,\ 3,\ y_1,\ y_2,\ \ldots,\ y_{2m-3}$$

of the identity, n being taken as $m-2$, by the respective numbers

$$2,\ 3,\ 4,\ 5,\ 6,\ \ldots,\ r-1,\ r+1,\ \ldots,\ 2m+1,\ r,$$

and rearranging.

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Similarly we obtain for the the respective coefficients of X_2X_r, X_3X_r, X_4X_r the expressions

$$(-1)^{r-2}[56\ldots(r-1)(r+1)\ldots(2m+1)]\quad[134\ldots(2m+1)],$$
$$(-1)^{r-1}[56\ldots(r-1)(r+1)\ldots(2m+1)]\quad[1245\ldots(2m+1)],$$
$$(-1)^{r}\quad[56\ldots(r-1)(r+1)\ldots(2m+1)][12356\ldots(2m+1)].$$

For the coefficient of X_r^2 we obtain

$$\begin{aligned}&[1256\ldots(r-1)(r+1)\ldots(2m+1)][3456\ldots(r-1)(r+1)\ldots(2m+1)]\\ -&[1356\ldots(r-1)(r+1)\ldots(2m+1)][2456\ldots(r-1)(r+1)\ldots(2m+1)]\\ +&[1456\ldots(r-1)(r+1)\ldots(2m+1)][2356\ldots(r-1)(r+1)\ldots(2m+1)]\\ =\ &[56\ldots(r-1)(r+1)\ldots(2m+1)][1234\ldots(r-1)(r+1)\ldots(2m+1)].\end{aligned}$$

Lastly, for the coefficient of X_rX_s, where r and s are both greater than 4, we have

$$\begin{aligned}(-1)^{r+s-6}\{&[1256\ldots(r-1)(r+1)\ldots(2m+1)][3456\ldots(s-1)(s+1)\ldots(2m+1)]\\ &-[1356\ldots(r-1)(r+1)\ldots(2m+1)][2456\ldots(s-1)(s+1)\ldots(2m+1)]\\ &+[1456\ldots(r-1)(r+1)\ldots(2m+1)][2356\ldots(s-1)(s+1)\ldots(2m+1)]\\ &+[1256\ldots(s-1)(s+1)\ldots(2m+1)][3456\ldots(r-1)(r+1)\ldots(2m+1)]\\ &-[1356\ldots(s-1)(s+1)\ldots(2m+1)][2456\ldots(r-1)(r+1)\ldots(2m+1)]\\ &+[1456\ldots(s-1)(s+1)\ldots(2m+1)][2356\ldots(r-1)(r+1)\ldots(2m+1)]\}\\ =(-1)^{r+s-6}\{&[56\ldots(r-1)(r+1)\ldots(2m+1)][1234\ldots(s-1)(s+1)\ldots(2m+1)]\\ &+[56\ldots(s-1)(s+1)\ldots(2m+1)][1234\ldots(r-1)(r+1)\ldots(2m+1)]\},\end{aligned}$$

by means of the identity of the type (28).

Thus, for the coefficient of

$$[56\ldots(r-1)(r+1)\ldots(2m+1)]$$

in our expression, we obtain

$$\begin{aligned}(-1)^{r-3}X_r\{&X_1[234\ldots(2m+1)]-X_2[134\ldots(2m+1)]+\ldots\\ &+(-1)^{r-1}X_r[123\ldots(r-1)(r+1)\ldots(2m+1)]+\ldots\\ &+(-1)^{s-1}X_s[123\ldots(s-1)(s+1)\ldots(2m+1)]+\ldots\\ &+(-1)^{2m}X_{2m+1}[123\ldots(2m)]\}\\ =&(-1)^{r-3}X_r[0123\ldots(2m+1)].\end{aligned}$$

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Thus our expression becomes

$$[0123\ldots(2m+1)]\{X_5[67\ldots(2m+1)]-X_6[578\ldots(2m+1)]+\ldots + (-1)^{2m-1}X_{2m+1}[56\ldots(2m)]\} = [0123\ldots(2m+1)][0567\ldots(2m+1)].$$

We thus arrive at the identity

$$\begin{aligned}&[01256\ldots(2m+1)][03456\ldots(2m+1)]\\ -&[01356\ldots(2m+1)][02456\ldots(2m+1)]\\ +&[01456\ldots(2m+1)][02356\ldots(2m+1)]\\ =&\ [0567\ldots(2m+1)]\ [0123\ldots(2m+1)] \quad \ldots\ldots(29).\end{aligned}$$

We thus see that the conditions of the type (25), and therefore also equations (24), require the evanescence of the expression

$$[0123\ldots(2m+1)].$$

Writing, for brevity,

$$u_{12} = [0345\ldots(2m+1)],$$
$$u_{13} = [0245\ldots(2m+1)],$$

and so on, we see that the relations of the type (25) can be written in the general form

$$u_{pq}u_{rs} - u_{pr}u_{qs} + u_{ps}u_{qr} = 0 \quad \ldots\ldots\ldots\ldots(30),$$

where p, q, r, s are four numbers, standing in their natural order, abstracted from the set

$$1, 2, 3, \ldots, 2m+1.$$

Further, we know that the u's are connected by the following set of differential relations.*

$$\left.\begin{aligned}&\frac{\partial u_{12}}{\partial x_2} - \frac{\partial u_{13}}{\partial x_3} + \ldots + (-1)^{2m+1}\frac{\partial u_{1(2m+1)}}{\partial x_{2m+1}} = m[234\ldots(2m+1)]\\ &\frac{\partial u_{12}}{\partial x_1} - \frac{\partial u_{23}}{\partial x_3} + \ldots + (-1)^{2m+1}\frac{\partial u_{2(2m+1)}}{\partial x_{2m+1}} = m[134\ldots(2m+1)]\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\frac{\partial u_{1(2m+1)}}{\partial x_1} - \frac{\partial u_{2(2m+1)}}{\partial x_2} + \ldots + (-1)^{2m+1}\frac{\partial u_{(2m)(2m+1)}}{\partial x_{2m}} = m[123\ldots(2m)]\end{aligned}\right\} \ldots(31).$$

* *Proc. Lond. Math. Soc.*, Vol. xxxi, p. 316.

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As before, we see that, should our superficial locus exist, we have to obtain it as a solution of the set of equations

$$\left.\begin{aligned} &u_{23}dx_1 + u_{13}dx_2 + u_{12}dx_3 = 0 \\ &u_{24}dx_1 + u_{14}dx_2 - u_{12}dx_4 = 0 \\ &u_{25}dx_1 + u_{15}dx_2 + u_{12}dx_5 = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &u_{2(2m+1)}dx_1 + u_{1(2m+1)}dx_2 + (-1)^{2m}u_{12}dx_{2m+1} = 0 \end{aligned}\right\} \ldots(32).$$

Further, taking account of the equations of the type (30), we know that the necessary conditions for equations (32) to constitute an exact system are of the type

$$u_{23}\left\{\frac{\partial u_{12}}{\partial x_2} - \frac{\partial u_{13}}{\partial x_3} + \ldots + (-1)^{2m+1}\frac{\partial u_{1(2m+1)}}{\partial x_{2m+1}}\right\}$$

$$- u_{13}\left\{\frac{\partial u_{12}}{\partial x_1} - \frac{\partial u_{23}}{\partial x_3} + \ldots + (-1)^{2m+1}\frac{\partial u_{2(2m+1)}}{\partial x_{2m+1}}\right\}$$

$$+ u_{12}\left\{\frac{\partial u_{12}}{\partial x_1} - \frac{\partial u_{23}}{\partial x_2} + \ldots + (-1)^{2m+1}\frac{\partial u_{3(2m+1)}}{\partial x_{2m+1}}\right\} = 0,$$

or

$$[0145\ldots(2m+1)][234\ldots(2m+1)]$$

$$-[0245\ldots(2m+1)][134\ldots(2m+1)]$$

$$+[0345\ldots(2m+1)][1245\ldots(2m+1)] = 0 \quad \ldots(33).$$

If we expand the expression on the left-hand side of equation (33), we obtain for the respective coefficients of X_1, X_2, X_3,

$$[45\ldots(2m+1)][234\ldots(2m+1)],$$

$$-[45\ldots(2m+1)][134\ldots(2m+1)],$$

$$[45\ldots(2m+1)][1245\ldots(2m+1)].$$

The coefficient of X_r, where $r > 3$, is

$$(-1)^{r-3}\{[145\ldots(r-1)(r+1)\ldots(2m+1)][234\ldots(2m+1)]$$

$$-[245\ldots(r-1)(r+1)\ldots(2m+1)][134\ldots(2m+1)]$$

$$+[345\ldots(r-1)(r+1)\ldots(2m+1)][1245\ldots(2m+1)]\}.$$

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In the manner in which the identity of the type (27) was obtained, we may readily establish the further identity

$$\begin{aligned}&[1y_1y_2\cdots y_{2m+1}][234y_1y_2\cdots y_{2m+1}]\\ &-[2y_1y_2\cdots y_{2m+1}][134y_1y_2\cdots y_{2m+1}]\\ &+[3y_1y_2\cdots y_{2m+1}][124y_1y_2\cdots y_{2m+1}]\\ &-[4y_1y_2\cdots y_{2m+1}][123y_1y_2\cdots y_{2m+1}]=0,\end{aligned}$$

and thus the above coefficient becomes

$$(-1)^{r-2}[45\ldots(2m+1)][123\ldots(r-1)(r+1)\ldots(2m+1)].$$

The expression on the left-hand side of equation (33) is thus reduced to

$$\begin{aligned}[45\ldots(2m+1)]\{&X_1[234\ldots(2m+1)]-X_2[134\ldots(2m+1)]\\ &+\&c.+(-1)^m X_{2m+1}[123\ldots(2m)]\}\\ &=[45\ldots(2m+1)][0123\ldots(2m+1)];\end{aligned}$$

and consequently all the conditions of the type (33) are satisfied by the evanescence of the expression

$$[0123\ldots(2m+1)].$$

Thus, in the first conditioned case, the equations (32) will possess $2m-1$ integrals, which will determine a definite $(n-2)$-tuple family of superficial loci.

If equations (24) are satisfied, equation (23) will then reduce to

$$d_{123\ldots(2m+1)}(x_1, x_2, \ldots, x_{2m+1}) = 0,$$

which indicates that the elements of the displacement symbolized by d_{2m+1} are linearly expressible in terms of those of the displacements symbolized by

$$d_1, d_2, \ldots, d_{2m}.$$

Now in the first conditioned case it is seen that the curves defined by equations (19) satisfy our Pfaffian equation. Further, from the above, the intersection of each pair of α's cuts a surface of our system in a curve. At this point, however, we find a divergence from the analogy to the case discussed in Part I.; for, if we treat equations (32) in a manner similar to that in which we treated equations (13), we obtain

$$u_{12}(X_1dx_1 + X_2dx_2 + \ldots + X_{2m+1}dx_{2m+1}) = 0.$$

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The surfaces of our system are thus solutions of equations (1). Further, combining equations (19) with equations (32), we obtain the equations of the type (33). Thus, though the curves defined by equations (19) lie upon the surface defined by equations (32), they are no longer the only curves lying on these surfaces that satisfy equation (1). It is, however, clear that, as the curves defined by equations (19) are still determinate, all the α's through a given point will meet the surface of our system, passing through the said point, in the same curve.

THE ASYMPTOTIC SOLUTION OF CERTAIN TRANSCENDENTAL EQUATIONS.

By G. H. Hardy, Trinity College, Cambridge.

I.

The equation $\phi(x)\, e^{G(x)} = \psi(x)$.

§ 1. M. BIRGER LINDGREN, in a recent paper,* has proved that if $\phi(x)$, $\psi(x)$, $G(x)$ are polynomials, the last of degree p, then the series formed by the $(-p)^{\text{th}}$ powers of the moduli of the roots of the equation

$$(1) \quad \phi(x)\, e^{G(x)} = \psi(x)$$

is divergent, provided that neither ϕ nor ψ vanishes identically. He has also proved that we can construct certain angles whose vertex is the origin, and whose magnitude σ is arbitrarily small, and choose R that all the roots whose distance from the origin is $> R$ lie within them.

I find that it is possible to obtain considerably more precise information concerning the roots by the elementary methods by which I obtained the asymptotic solution of the simpler equations

$$e^x = P(x), \quad \sin x = P(x)$$

in two papers which appeared recently in the *Messenger of Mathematics* (Vol. XXXI., p. 161, and Vol. XXXII., p. 36).

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* *Sur la fonction entière* $e^{K(s)} P_1(s) + P(s)$ (*Bihang till K. Svenska Vet. Akad. Handlingar*, 28, Afd. I., No. 9).

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§ 2. I suppose for the present that $p \geqq 2$. It is clear that we can, without loss of generality, suppose that the polynomial $G(x)$ is

$$ix^p + (a_1 + ib_1)\, x^{p-1} + \ldots,$$

where neither a_1 nor b_1 is zero. For if this is not the case we have only to transform the equation (1) by a suitably chosen linear equation

$$x = Ay + B.$$

Moreover, we may express the rational fraction

$$\frac{\phi(x)}{\psi(x)}$$

in the form

$$(c + id)\, x^q \left(1 + \frac{\rho}{x}\right),$$

where ρ is a function of x which is less than a constant H for all values of x whose modulus is $> R$.

If then

$$x \equiv \xi + i\eta = re^{i\theta},$$

$$a_1 + ib_1 = \alpha_1 e^{i\beta_1},$$

$$c + id = \gamma e^{i\mu},$$

we obtain, on equating the moduli of the two sides of (1),

$$(2) \quad \exp.\left[-r^p \sin p\theta + \alpha_1 r^{p-1} \cos\{\beta_1 + (p-1)\,\theta\} + \ldots\right]$$

$$= \gamma r^q \left(1 + \frac{\rho}{r}\right),$$

where ρ now is a function of r, whose modulus $< H$ for all values of $r > R$. I may remark that in what follows I shall generally use ρ to signify a quantity whose modulus $< H$ for values of r greater than a certain value, and ϵ for a quantity whose modulus may be made as small as we please by sufficiently increasing r; these quantities ρ, ϵ will of course not be the same in different equations.

It is evident that if $|\sin p\theta| > \eta$ (however small be the positive quantity η) we can choose R so that the equation (2) is impossible for $r > R$. Now the inequality

$$|\sin p\theta| < \eta$$

implies $$m\pi - p\sigma < p\theta < m\pi + p\sigma,$$

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where m is an integer and $p\sigma = \sin^{-1}\eta$. Hence θ satisfies one of the sets of inequalities

$$\frac{m\pi}{p} - \sigma < \theta < \frac{m\pi}{p} + \sigma \quad (m = 0,\ 1,\ \ldots,\ p-1).$$

As a matter of fact this consequence holds for the more general equation in which ϕ, ψ are any integral functions whose apparent order is $< p$; or, more generally still, for any function whose modulus

$$< e^{\epsilon r^p},$$

however small be ϵ, for all values of $r > R$.

§ 3. I shall now consider more particularly the roots of (1) (if any) for which R is large and $|\theta|$ small. For such roots

$$\left|\frac{\xi}{\eta}\right|$$

is large; and (2) may be written in the form

$$(3) \quad \exp.\{-p\eta\xi^{p-1} + \ldots + a_1\xi^{p-1} + \ldots\} = \gamma(\xi^2 + \eta^2)^{\frac{1}{2}q}\left(1 + \frac{\rho}{r}\right)$$

$$= \gamma\xi^q\left(1 + \frac{\rho}{\xi}\right),$$

and it is easy to see that this is only possible if η is very nearly equal to

$$\frac{a_1}{p}.$$

Taking logarithms and arranging the result in powers of ξ we obtain

$$-(p\eta - a_1)\,\xi^{p-1} + \{-(p-1)\,b_1\eta + a_2\}\,\xi^{p-2} + \ldots$$

$$-\gamma q \log\xi - \frac{\rho}{\xi} = 0.$$

If ξ is large and η not large the sign of the left-hand obviously depends on that of

$$-(p\eta - a_1);$$

and its differential coefficient with respect to η is

$$-p\xi^{p-1} + \ldots,$$

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which is negative. Hence it vanishes once and once only for a value of η in the neighbourhood of $\eta = \frac{a_1}{p}$.

Suppose that the result of substituting $\frac{a_1}{p}$ for η in the coefficient of ξ^{p-2} is a quantity A differing from 0 (as will generally be the case). Then it is clear that an approximation for $\eta - \frac{a_1}{p}$ is given by

$$\eta - \frac{a_1}{p} = \frac{A}{p\xi}.$$

Again
$$\frac{d\eta}{d\xi} = -\frac{P}{Q},$$

where*

$$P = -p(p-1)\left(\eta - \frac{a_1}{p}\right)\xi^{p-2} + (p-2)\{-(p-1)b_1\eta + a_2\}\xi^{p-3} + \ldots,$$

$$Q = -p\xi^{p-1} + \ldots.$$

The dominant term in P is

$$\{-(p-1) + p - 2\}A\xi^{p-2} = -A\xi^{p-2},$$

so that $d\eta/d\xi$ is small (of order $1/\xi^2$) and has the sign of $-A$.

§ 4. We have now to consider the equation obtained by expressing the fact that the amplitudes of the two sides of the equation

$$e^{G(x)} = \frac{\phi(x)}{\psi(x)}$$

differ by a multiple of 2π. It is

$$(4)\quad \xi^p + b_1\xi^{p-1} + \left\{-\frac{p(p-1)}{1.2}\eta^2 + (p-1)a_1\eta + b_2\right\}\xi^{p-2} + \ldots - (2k\pi + q\theta + \mu) - \frac{\rho}{\xi} = 0,$$

where k is an integer, which must be large and positive if ξ is large and positive and η not large.

The differential coefficient with respect to ξ of the left-hand side of (4) is

$$p\xi^{p-1} + \ldots,$$

* It is easy to see that the partial differential coefficients of the small quantity $\frac{\rho}{\xi}$ are also small.

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which is certainly positive for large values of ξ. Hence the left-hand side vanishes, when η is given any value not far from $\frac{a_1}{p}$, for one and only one large value of ξ. This value of ξ is determined by an equation

$$\xi^p\left(1+\frac{b_1}{\xi}+\frac{\rho}{\xi^2}\right)=2k\pi+\mu+\frac{\rho}{\xi},$$

or
$$\xi\left(1+\frac{b}{p\xi}+\frac{\rho}{\xi^2}\right)=(2k\pi)^{\frac{1}{p}}+\frac{\rho}{k},$$

or
$$\xi=\left\{(2k\pi)^{\frac{1}{p}}+\frac{\rho}{k}\right\}\left(1-\frac{b_1}{p\xi}+\frac{\rho}{\xi^2}\right)$$

$$=(2k\pi)^{\frac{1}{p}}-\frac{b_1}{p}+\epsilon.$$

Finally, along the part of the curve defined by (4), which lies in the part of the plane under consideration,

$$\frac{d\eta}{d\xi}=-\frac{P}{Q},$$

where

$$P=p\xi^{p-1}+\ldots,$$

$$Q=-\{p(p-1)\eta-(p-1)a_1\}\xi^{p-2}+\ldots$$

$$=-(p-1)A\xi^{p-2}+\ldots,$$

so that $d\eta/d\xi$ is large and has the sign of A.

If we compare the results of this and the preceding section, we see that in this part of the plane the equations

$$\left|e^{G(x)}\right|=\left|\frac{\phi(x)}{\psi(x)}\right|$$

and
$$\text{amp. } e^{G(x)}=\text{amp. }\frac{\phi(x)}{\psi(x)}+2k\pi$$

define, the first a single continuous curve asymptotic to the line

$$\eta=\frac{a_1}{p},$$

the second an infinite series of curves each of which cuts the former in a single point. And these points are roots of the

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equation (1). Thus *the equation* (1) *possesses a series of roots which tend asymptotically to the points*

$$\sqrt[p]{(2k\pi)} + \frac{-b_1 + ia_1}{p}.$$

We assumed that the quantity

$$A = a_2 - \frac{(p-1)\,a_1 b_1}{p}$$

did not vanish. If it vanishes the investigation is slightly more complicated, but the result is the same; and of course we can always ensure that it does not by making a suitable preliminary transformation of the type

$$x = \eta + B.$$

The equation (1) has also p other sets of roots which may be determined similarly. If we make the substitution

$$x = e^{\frac{2m\pi i}{p}} y,$$

it is transformed into an equation of similar form which possesses a series of roots lying in the direction $\theta = 0$. These give the roots of the original equation which lie in the direction

$$\theta = \frac{2m\pi}{p}.$$

§ 5. To verify our results let us calculate those roots of

$$(5) \quad e^{ix^{2m} + (a+ib)x^m + \alpha} = \beta,$$

which lie at a great distance in the positive direction of the real axis. They are given by

$$x^{2m} + (b - ia)\,x^m - 2k\pi + H = 0,$$

where H is a constant and k a large positive integer. We must take

$$x^m = \tfrac{1}{2}(ia - b) + \sqrt{\{2k\pi - H + \tfrac{1}{2}(ia - b)^2\}}$$
$$= \sqrt{(2k\pi)} + \tfrac{1}{2}(ia - b) + \epsilon$$

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(ϵ being small when k is large). If $m=1$,

$$(6) \quad x=\sqrt{(2k\pi)}+\tfrac{1}{2}(ia-b)+\epsilon,$$

but if $m>1$,

$$(7) \quad x=\sqrt[m]{(2k\pi)}+\epsilon,$$

the result in either case agreeing with the preceding section. We might apply this method to the more general equation

$$e^{G(x)}=\beta.$$

§ 6. We have so far supposed $p\geqq 2$. If $p=1$ the results are of a different character. In my former papers I investigated the roots of

$$e^x=P(x),$$

or (what is the same thing) those of

$$e^{-x}=P(x).$$

I, however, imposed certain unnecessary restrictions on the coefficients of $P(x)$. The general asymptotic solution is as follows, if

$$P(x)=(\alpha+i\beta)\,x^n+\ldots,$$

$$\left.\begin{aligned}&\xi=2k\pi+\mu,\\&\eta=\log\sqrt{(\alpha^2+\beta^2)}+n\log|2k\pi|\\&\qquad(k=-\infty,\ \ldots,\ +\infty)\end{aligned}\right\}(n \text{ even}),$$

$$\left.\begin{aligned}&\xi=2k\pi+\mu\\&\eta=\log\sqrt{(\alpha^2+\beta^2)}+n\log 2k\pi\\&\qquad(k=\ldots+\infty)\\&\xi=2k\pi+\mu'\\&\eta=\log\sqrt{(\alpha^2+\beta^2)}+n\log|2k\pi|\\&\qquad(k=-\infty\ \ldots)\end{aligned}\right\}(n \text{ odd}),$$

μ, μ' being angles between $-\pi$ and $+\pi$, and such that

$$\cos\mu:\sin\mu::1::\alpha:-\beta:\sqrt{(\alpha^2+\beta^2)},$$

$$\cos\mu':\sin\mu'::1::-\alpha:\beta:\sqrt{(\alpha^2+\beta^2)}.$$

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It is quite easy to write down the more general solution of

$$\psi(x)\, e^{-ix} = \phi(x).$$

I shall not trouble to do this, but I wish to point out the essential difference between the solution of (1) in the cases of $p > 1$ and $p = 1$. In the former case the solution is independent of the form of ϕ and ψ; in the latter it is not. This is not in the least difficult to understand. For it is clear that the divergence between the types of the series of roots of

$$\phi(x)\, e^{G(x)} = \psi(x)$$

for different forms of ϕ and ψ is likely to be lessened by an increase in the disparity between the rates of growth of the transcendental function $e^{G(x)}$ and the polynomials.

§ 7. The equation

$$\psi(x) \sin G(x) = \phi(x)$$

may be solved similarly. In the simple case of

$$\sin x = P(x) = (\alpha + i\beta)\, x^n + \ldots,$$

the solution is

$$\left.\begin{aligned} &\xi = 2k\pi + \lambda \\ &\eta = \log 2\sqrt{(\alpha^2+\beta^2)} + n \log |\, 2k\pi \,| \\ &\qquad (k = -\infty \ \ldots + \infty) \end{aligned}\right\} (n \text{ even}),$$

$$\left.\begin{aligned} &\xi = 2k\pi + \lambda \\ &\eta = \log 2\sqrt{(\alpha^2+\beta^2)} + n \log 2k\pi \\ &\qquad (k = \ldots + \infty) \\ &\xi = 2k\pi + \lambda' \\ &\eta = \log 2\sqrt{(\alpha^2+\beta^2)} + n \log |\, 2k\pi \,| \\ &\qquad (k = -\infty \ \ldots) \end{aligned}\right\} (n \text{ odd}),$$

λ, λ' being angles between $-\pi$ and $+\pi$, and such that

$$\cos\lambda : \sin\lambda :: 1 :: \alpha : \beta : \sqrt{(\alpha^2+\beta^2)},$$

$$\cos\lambda' : \sin\lambda' :: 1 :: -\alpha : -\beta : \sqrt{(\alpha^2+\beta^2)}.$$

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§ 8. I may add that in many cases we can find similar solutions for the more general equations

$$e^{-ix} = f(x), \quad \sin x = f(x),$$
$$\ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots,$$

in which $f(x)$ is any integral function which satisfies certain conditions, of which the most obvious and important is that

$$|f(x)| < e^{r^\rho} \quad (\rho < 1),$$

for all sufficiently great values of r.

II.

The equation $e^{ax} \sin bx = P(x)$.

§ 9. I shall now consider another equation,

$$(8) \quad e^{ax} \sin bx = P(x),$$

where a, b are for simplicity supposed real and positive, and $P(x)$ is an arbitrary polynomial. The solution of this equation presents features which which have been absent from those which I have considered before. These peculiarities appear quite clearly in the particular equation

$$(9) \quad e^{ax} \sin bx = c$$

(where c is any constant but zero), which I shall therefore consider first.

The equation $e^{ax} \sin bx = c$.

§ 10. If a, b are real and positive, $c = \gamma + i\delta$, $x = \xi + i\eta$, we obtain at once, on equating real and imaginary parts,

$$e^{a\xi - b\eta} \cos(a\eta + b\xi) - e^{a\xi + b\eta} \cos(a\eta - b\xi) = -2\delta,$$
$$e^{a\xi - b\eta} \sin(a\eta + b\xi) - e^{a\xi + b\eta} \sin(a\eta - b\xi) = 2\gamma.$$

From these two equations, or by equating the moduli of the two sides of (9), we deduce

$$\cosh 2b\eta - \cos 2b\xi = 2(\gamma^2 + \delta^2) e^{-2a\xi}.$$

It follows that if ξ is large and positive

$$\cosh 2b\eta - \cos 2b\xi$$

is small. This can only be the case if η is small and $b\xi$ nearly equal to a multiple of π.

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Thus the roots for which ξ is large and positive approach asymptotically to some or all of the points

$$\frac{n\pi}{b},$$

i.e. to the zeroes of $e^{ax}\sin bx$. On the other hand if ξ is large and negative η is large and

$$e^{2b|\eta|} = 4(\gamma^2+\delta^2)e^{-2a\xi}(1+\epsilon),$$

where ϵ is small, and in fact less than a constant multiple of

$$e^{-4b|\eta|}.$$

Hence

$$|\eta| = -\frac{a\xi}{b} + \frac{1}{2b}\log 4(\gamma^2+\delta^2) + \epsilon,$$

where ϵ satisfies the same condition as before.

First, suppose $\eta > 0$. Then

$$e^{b\eta} = 2\sqrt{(\gamma^2+\delta^2)}\,e^{-a\xi}(1+\epsilon),$$

and also

$$e^{b\eta}\cos(a\eta - b\xi) = \quad 2\delta e^{-a\xi}(1+\epsilon),$$

$$e^{b\eta}\sin(a\eta - b\xi) = -2\gamma e^{-a\xi}(1+\epsilon),$$

so that

$$\cos(a\eta - b\xi) = \frac{\delta}{\sqrt{(\gamma^2+\delta^2)}}(1+\epsilon),$$

$$\sin(a\eta - b\xi) = \frac{-\gamma}{\sqrt{(\gamma^2+\delta^2)}}(1+\epsilon).$$

Let λ be the angle between $-\pi$ and π, such that

$$\cos\lambda : \sin\lambda : 1 :: \delta : -\gamma : \sqrt{(\gamma^2+\delta^2)}.$$

Then $$a\eta - b\xi = 2p\pi + \lambda + \epsilon.$$

But, since

$$b\eta + a\xi = \tfrac{1}{2}\log 4(\gamma^2+\delta^2) + \epsilon,$$

it follows that

$$\begin{cases} (a^2+b^2)\,\xi = a\log 2|c| - (2p\pi+\lambda)\,b + \epsilon, \\ (a^2+b^2)\,\eta = b\log 2|c| + (2p\pi+\lambda)\,a + \epsilon. \end{cases}$$

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The roots for which ξ is large and negative, and $\eta > 0$, tend to the points

$$\frac{1}{a^2+b^2}[a \log 2|c| - (2p\pi + \lambda) b + i\{b \log 2|c| + (2p\pi + \lambda) a\}],$$

where p is a large positive integer.

On the other hand, if η is negative, we obtain

$$e^{-b\eta} = 2\sqrt{(\gamma^2+\delta^2)}\, e^{-a\xi}(1+\epsilon),$$

$$e^{-b\eta}\cos(a\eta + b\xi) = -2\delta e^{-a\xi}(1+\epsilon),$$

$$e^{-b\eta}\sin(a\eta + b\xi) = 2\gamma e^{-a\xi}(1+\epsilon),$$

$$\cos(a\eta + b\xi) = \frac{-\delta}{\sqrt{(\gamma^2+\delta^2)}}(1+\epsilon),$$

$$\sin(a\eta + b\xi) = \frac{\gamma}{\sqrt{(\gamma^2+\delta^2)}}(1+\epsilon),$$

$$a\eta + b\xi = -2p\pi - \lambda + \epsilon,$$

$$-b\eta + a\xi = \tfrac{1}{2}\log 4(\gamma^2+\delta^2) + \epsilon,$$

$$\begin{cases} (a^2+b^2)\xi = a \log 2|c| - (2p\pi+\lambda) b + \epsilon, \\ (a^2+b^2)\eta = -b \log 2|c| - (2p\pi+\lambda) a + \epsilon. \end{cases}$$

Thus the points to which these roots tend are the conjugates of the former set.

Finally, we observe that η can only be large if ξ is large and negative. Thus all possible roots at a large distance from the origin are included in one or other of the three sets which we have investigated.

The question remains: does one and only one root correspond to each of the points of the three sets to which we have proved that the roots asymptotically tend?

In order to settle this question we begin by tracing the curves

$$|e^{ax}\sin bx| = |c|,$$

or

$$\cosh 2b\eta - \cos 2b\xi = 2(\gamma^2+\delta^2)e^{-2a\xi},$$

in so far as they lie in the part of the plane for which ξ is large and positive.

The preceding analysis shows that all points of any such curve lie very near to one or other of the points $\frac{n\pi}{b}$.

Let

$$\cos 2b\xi + 2(\gamma^2+\delta^2)e^{-2a\xi} = \theta(\xi).$$

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Then from

$$\cosh 2b\eta = \theta(\xi),$$

we deduce

$$e^{2b\eta} = \theta(\xi) \pm \sqrt{[\{\theta(\xi)\}^2 - 1]},$$

$$e^{-2b\eta} = \theta(\xi) \mp \sqrt{[\{\theta(\xi)\}^2 - 1]}.$$

In order that these equations should be satisfied we must have $\theta(\xi) > 1$, and then we must take the upper or lower sign according as η is positive or negative. Thus to each value of ξ which lies in the small range of values of ξ on each side of $\frac{n\pi}{b}$, for which $\theta(\xi) > 1$, correspond two and only two values of η which coalesce in 0 for the values of ξ for which $\theta(\xi) = 1$. It follows that the curves consist of an infinite number of simple closed curves surrounding each of the points $\frac{n\pi}{b}$, and shrinking up into these points as n tends to ∞. And from this it follows by a well known theorem that for large values of n there is one and only one root of the equation (9) near each of the points $\frac{n\pi}{b}$, these points giving the roots of (9) asymptotically.

It remains to consider the other two sets of zeroes associated with the essential singularity.

Suppose first that ξ is large and negative, η large and positive, and

$$|e^{ax} \sin bx| = |c|.$$

Then $$b\eta + a\xi = \log 2|c| + \epsilon,$$

so that the part of the curve at a great distance from 0 in this direction is asymptotic to the straight line

$$b\eta + a\xi = \log 2|c|.$$

Moreover, since

$$\cosh 2b\eta = \cos 2b\xi + 2(\gamma^2 + \delta^2) e^{-2a\xi},$$

$$2b \sinh 2b\eta \frac{d\eta}{a\xi} = -2b \sin 2b\xi - 4a(\gamma^2 + \delta^2) e^{-2a\xi},$$

so that $\frac{d\eta}{d\xi}$ is ultimately of constant (negative) sign, being in fact nearly $= -\frac{a}{b}$.

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Now suppose

$$am.c. = \tau,$$

and consider the curves

$$am.e^{ax}\sin bx = \tau,$$

whose intersections with the former curve give the zeroes This gives

$$-\tan\tau = -\frac{\delta}{\gamma} = \frac{e^{b\eta}\cos(a\eta - b\xi) - e^{-b\eta}\cos(a\eta + b\xi)}{e^{b\eta}\sin(a\eta - b\xi) - e^{-b\eta}\sin(a\eta + b\xi)}$$

$$= \frac{\tanh b\eta + \tan b\xi \tan a\eta}{\tanh b\eta \tan a\eta - \tan b\xi},$$

$$\tan b\xi = \tanh b\eta \cot(\tau - a\eta).$$

This equation represents a family of parallel curves, the distance between them (measured parallel to ξ) being $\frac{\pi}{b}$. In the region of the plane which we are considering they are nearly coincident with the lines

$$a\eta - b\xi = p\pi + \tau - \tfrac{1}{2}\pi = p\pi + \lambda.$$

Of these lines those for which p is even correspond to $am.e^{ax}\sin bx = \tau$, those for which p is odd to

$$am.e^{ax}\sin bx = \tau + \pi,$$

and it is easy to see that when η is large $\frac{d\eta}{d\xi}$ is certainly positive along them.

It follows that each of the lines cuts the former curve $|e^{ax}\sin bx| = |c|$ in one and only one point for which η is large, and hence that one and only one root of $e^{ax}\sin bx = c$ is associated with each of the second set of points previously determined.

Similarly for the third set. The roots of $e^{ax}\sin bx = c$ for which $|x|$ is large are therefore completely represented asymptotically by the three sets of points.

§ 11. The result of the last paragraph enables us to give an affirmative answer to the following question.

Is it possible that, if $a_n(c)$ is the n^{th} zero of

$$F(x) - c$$

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(the zeroes being arranged in order of moduli), *then*

$$\lim_{n=\infty} \left| \frac{a_n(c_1)}{a_n(c_2)} \right| = l,$$

where l *is not* $= 1$?

For consider the three sets of zeroes of (9), c being not $= 0$. The first set tends to the points

$$\frac{p\pi}{b},$$

the second set to the points

$$\frac{-b+ia}{a^2+b^2} 2p\pi + \delta,$$

and the third to the points

$$\frac{-b-ia}{a^2+b^2} 2p\pi + \delta',$$

δ, δ' being quantities which do not increase indefinitely with n. The moduli of the three sets of zeroes are roughly

$$\frac{p\pi}{b}, \quad \frac{2p\pi}{\sqrt{(a^2+b^2)}}, \quad \frac{2p\pi}{\sqrt{(a^2+b^2)}},$$

and the number of zeroes within a large circle of radius R is roughly

$$\frac{bR}{\pi} + \frac{\sqrt{(a^2+b^2)}\,R}{\pi},$$

whereas for $e^{ax}\sin bx$ this number is

$$\frac{2bR}{\pi}.$$

If, *e.g.* $a = b\sqrt{3}$, the moduli of the p^{th} zeroes of the three sets are roughly

$$\frac{p\pi}{b}, \quad \frac{p\pi}{b}, \quad \frac{p\pi}{b},$$

so that the modulus of the $3p^{\text{th}}$ zero is roughly $\frac{p\pi}{b}$, whereas for $e^{ax}\sin bx$ the modulus of the $2p^{\text{th}}$ zero is roughly $\frac{p\pi}{b}$.

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§ 12. M. Borel* has suggested that the following theorem is probably true.

If $\alpha_n(c)$ is the modulus of the n^{th} zero of $F(x)-c$, and one particular value of c is excluded from consideration, then

$$\lim. \frac{\alpha_n(c_1)}{\alpha_n(c_2)} = 1.$$

It is easy to verify that the example which we have been discussing obeys this theorem. In this case, of course, $c=0$ is the particular value of c; it is what we may call the Picard case, if we interpret this expression in a more general sense. In all the particular cases which I have examined there is at most one such exceptional case. At the same time it seems to me highly improbable that the general theorem above stated is true. But there can be no doubt that is true for large classes of functions; but it seems exceedingly difficult even to verify it except when we know at least some asymptotic expression of the function under consideration. It seems to me clear from the evidence of the particular cases which I have studied that, *for large classes of functions*, one can go much further and assert that *the actual zeroes* $a_n(c_1)$, $a_n(c_2)$—not merely their moduli—can be arranged in a number of groups such that *within each group*

$$\lim. \frac{a_n(c_1)}{a_n(c_2)} = 1.$$

It is, indeed, quite easy to see in a general way why this should be so; but to formulate precise conditions under which it is so and to give a formal proof for functions satisfying these conditions is quite another matter.

The equation $e^{az}\sin bx = P(x)$.

§ 13. I shall now consider the more general equation (8). If

$$e^{az}\sin bx = P(x) = a_n x^n + \ldots + a_1 x + a_0,$$

we find at once that

$$\cosh 2b\eta - \cos 2b\xi = 2\,|\,P(x)\,|\,e^{-2a\xi}.$$

If ξ is large and positive, the right-hand is certainly small, unless η is very large compared with ξ, in which case the equation is plainly impossible. If ξ is large and negative,

* *Leçons sur les fonctions entières*, p. 100.

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$|\eta|$ is large. If $|\eta|$ is large, ξ is large. We have thus three sets of zeroes to consider corresponding exactly to the three cases of § 11.

(i) ξ large and positive. Then $\cosh 2b\eta - \cos 2b\xi$ is small, and x is close to one of the points

$$\frac{n\pi}{b},$$

as in § 11.

(ii) ξ large and negative, η large and positive. Let

$$P(x) = \phi(\xi, \eta) + i\psi(\xi, \eta),$$

$$a_s = |a_s| e^{i\theta_s} \quad (s = 0, 1, \ldots, n),$$

$$a + ib = \sqrt{(a^2 + b^2)}\, e^{i\chi}.$$

Then

$$e^{a\xi - b\eta}\cos(a\eta + b\xi) - e^{a\xi + b\eta}\cos(a\eta - b\xi) = -2\psi,$$

$$e^{a\xi - b\eta}\sin(a\eta + b\xi) - e^{a\xi + b\eta}\sin(a\eta - b\xi) = 2\phi,$$

$$\cosh 2b\eta - \cos 2b\xi = 2(\phi^2 + \psi^2)\, e^{-2a\xi},$$

from which it follows that

$$\eta = -\frac{a\xi}{b}(1 + \epsilon),$$

and hence that

$$(\phi^2 + \psi^2) = |a_n|^2 (\xi^2 + \eta^2)^n (1 + \epsilon)$$

$$= |a_n|^2 \xi^{2n} \left(1 + \frac{a^2}{b^2}\right)^n (1 + \epsilon),$$

$$e^{b\eta} = 2|a_n| \left(1 + \frac{a^2}{b^2}\right)^{\frac{1}{2}n} |\xi|^n e^{-a\xi}(1 + \epsilon),$$

$$b\eta = -a\xi + n\log|\xi| + \log\left\{2|a_n|\left(1 + \frac{a^2}{b^2}\right)^{\frac{1}{2}n}\right\} + \epsilon.$$

Also

$$P(x) = |a_n| e^{i\theta_n} \xi^n \left(1 - \frac{ia}{b}\right)^n (1 + \epsilon)$$

$$= |a_n| \left(1 + \frac{a^2}{b^2}\right)^{\frac{1}{2}n} \xi^n \exp.\left[\{\theta_n + n(\chi - \tfrac{1}{2}\pi)\}\, i\right](1 + \epsilon),$$

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so that

$$e^{b\eta}\cos(a\eta - b\xi) = 2|a_n|\left(1+\frac{a^2}{b^2}\right)^{\frac{1}{2}n}\xi^n e^{-a\xi}\sin\{\theta_n + n(\chi - \tfrac{1}{2}\pi)\}(1+\epsilon),$$

$$e^{b\eta}\sin(a\eta - b\xi) = -2|a_n|\left(1+\frac{a^2}{b^2}\right)^{\frac{1}{2}n}\xi^n e^{-a\xi}\cos\{\theta_n + n(\chi - \tfrac{1}{2}\pi)\}(1+\epsilon),$$

or, if we suppose n even,

$$\cos(a\eta - b\xi) = \sin\{\theta_n + n(\chi - \tfrac{1}{2}\pi)\}(1+\epsilon),$$

$$\sin(a\eta - b\xi) = -\cos\{\theta_n + n(\chi - \tfrac{1}{2}\pi)\}(1+\epsilon),$$

$$a\eta - b\xi = 2p\pi + \lambda + \epsilon,$$

where $$\lambda = \theta_n - \tfrac{1}{2}\pi + n(\chi - \tfrac{1}{2}\pi).$$

Also $$b\eta + a\xi = n\log|\xi| + \log H + \epsilon,$$

if $$H = 2|a_n|\left(1+\frac{a^2}{b^2}\right)^{\frac{1}{2}n}.$$

Therefore

$$(a^2+b^2)\eta = b(n\log|\xi| + \log H) + a(2p\pi + \lambda) + \epsilon,$$

$$(a^2+b^2)\xi = a(n\log|\xi| + \log H) - b(2p\pi + \lambda) + \epsilon.$$

The dominant term on the right of the expression for $(a^2+b^2)\xi$ is $-2bp\pi$, and it follows that

$$\log|\xi| = \log(2bp\pi) - \log(a^2+b^2) + \epsilon.$$

Substituting in the former equations, we obtain finally

$$(a^2+b^2)\eta = a(2p\pi + \lambda) + b\{n\log 2p\pi - n\log\sqrt{(a^2+b^2)} + \log 2|a_n|\} + \epsilon,$$

$$(a^2+b^2)\xi = -b(2p\pi + \lambda) + a\{n\log 2p\pi - n\log\sqrt{(a^2+b^2)} + \log 2|a_n|\} + \epsilon.$$

If we make $n=0$, $a_n = c$, these equations reduce to those obtained in § 11.

In the same way we may determine the corresponding points for which η is negative. The case in which n is odd may be treated similarly. But it is hardly worth while to work out all the different cases in detail.

We have still to prove that one, and only one, zero is associated with each of these points. But before entering on

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this discussion the following remark should be made. The character of the roots of $e^{ax}\sin bx = 0$ is completely altered by writing c for 0 on the right-hand side of the equation, the disturbance of a root being of the same order of greatness as the root itself. But the further disturbance produced by writing, say, $c+x$ instead of c is quite analogous to the disturbance produced in the roots of $\sin x = 0$ by writing x instead of 0 on the right-hand side.

It is very easy to see how this must be the case. Draw the lines

$$\frac{\eta}{\xi} = -\frac{a}{b} + k, \quad \frac{\eta}{\xi} = -\frac{a}{b} - k,$$

where k is any small fixed positive quantity. Then in the region between the first of these and the positive part of the real axis, $|e^{ax}\sin bx|$ grows exponentially with $|x|$, while in the region between the second line and the negative part of the real axis it decreases exponentially with $|x|$. Along the real axis, it on the whole increases exponentially, but decreases with very great rapidity near the points $\frac{n\pi}{b}$. Now we have been considering points for which it is comparable with some power of $|x|$, and it is clear that these must lie ultimately very near to the zeroes $\frac{n\pi}{b}$, or on curves which ultimately lie entirely between the lines

$$\frac{\eta}{\xi} = -\frac{a}{b} \pm k,$$

and this however small be k.

We proceed now to the proof that one, and only one, zero is associated with each of the points which we have determined. This is easily shown for the points $\frac{n\pi}{b}$, for it is clear that when n is large the curves

$$\left|\frac{e^{ax}\sin bx}{P(x)}\right| = K$$

(where K is a positive constant > 1), which lie in this part of the plane, form a series of small closed curves surrounding these points. There are therefore as many roots of

$$e^{ax}\sin bx = P(x)$$

as of

$$e^{ax}\sin bx = 0$$

within this contour, that is to say one.

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We consider next the part of the curve

$$\left| \frac{e^{ax} \sin bx}{P(x)} \right| = 1,$$

for which ξ is large and negative, η large and positive. Along this

$$\cosh 2b\eta - \cos 2b\xi = 2\,(\phi^2 + \psi^2)\, e^{-2a\xi}.$$

The curve is asymptotic to

$$b\eta + a\xi = n \log |\,\xi\,| + \log \left\{ 2\,|\,a_n\,| \left(1 + \frac{a^2}{b^2}\right)^{\frac{1}{2}n} \right\},$$

and, moreover, since

$$\sinh 2b\eta \,.\, 2bd\eta + \sin 2b\xi \,.\, 2bd\xi$$
$$= -\,4a\,(\phi^2 + \psi^2)\, e^{-2a\xi} d\xi + 2e^{-2a\xi} d\,(\phi^2 + \psi^2),$$

and

$$\frac{d\,(\phi^2 + \psi^2)}{\phi^2 + \psi^2}$$

is ultimately very small, $\dfrac{d\eta}{d\xi}$ is ultimately of constant sign $\left(\text{being nearly} = -\dfrac{a}{b}\right)$.

Again, as we go along this curve $am\,.\,P(x)$ is ultimately always very nearly equal to $\theta_n + n\,(\chi - \frac{1}{2}\pi) = \mu$, say. The amplitude of $e^{ax} \sin bx$ is, as we saw, equal to μ along certain curves nearly coinciding in this part of the plane with the lines

$$a\eta - b\xi = 2p\pi - \tfrac{1}{2}\pi + \mu,$$

which cut the curve $|\,e^{ax} \sin bx\,| = |\,P(x)\,|$ each in one point nearly coinciding with one of the points to which the zeroes were shown to tend. At one of these points of intersection the amplitude of

$$\frac{e^{ax} \sin bx}{P(x)}$$

is nearly equal to a multiple of 2π.

Let A_p, A_{p+1} be two of these points. Then $am\,.\,e^{ax} \sin bx$ continually varies in the same sense as we go along the curve from A_p to A_{p+1}. Moreover, so does

$$am\,.\,\frac{e^{ax} \sin bx}{P(x)};$$

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for if it were stationary we should meet with a zero of

$$\frac{d}{dx}\frac{e^{ax}\sin bx}{P(x)} = \frac{e^{ax}}{P(x)}\left\{a\sin bx + b\cos bx - \frac{P'(x)\sin bx}{P(x)}\right\}.$$

But the leading term in the bracket is

$$\tfrac{1}{2}\{(b+ai)\,e^{b\eta - b\xi i}\},$$

so that this is impossible. The amplitude therefore varies in the same sense from A_p to A_{p+1}, and its change is nearly equal that of $am.e^{ax}\sin bx$, *i.e.* nearly equal to 2π. And we can prove in just the same manner that at a point half-way between A_p and A_{p+1} it has changed by nearly π, and at a point half-way between A_{p+1} and A_{p+2} by nearly 3π. There is a point between these two points at which it is exactly equal to an even multiple of π. This is a root of

$$e^{ax}\sin bx = P(x).$$

We have thus arrived at the desired conclusion.

The equation $\Pi_\rho(x) = P(x)$.

§ 14. I have carried out a similar investigation for the equations

$$(10)\quad \Pi_\rho(x) = c,$$

$$(11)\quad \Pi_\rho(x) = P(x),$$

where $P(x)$ is as before a polynomial, and $\Pi_\rho(x)$ is Mr. Barnes' function

$$\Pi_\rho(x) = \prod_1^\infty\left(1+\frac{x}{n^\rho}\right)\quad(\rho>1).$$

It would take too long now to give the details of my investigations, but I take this opportunity of stating my results, as M. Wiman has in a recent note* indicated results somewhat similar to mine. I have only seen the abstract of M. Wiman's memoir, and as far as I am aware it has not been published in full, so that it is impossible for me to say exactly what is the relation between his results and mine; but I am inclined to think that, approaching the matter from a different

* A. Wiman, "*Sur le genre de la fonction dérivée d'une fonction entière et sur le cas d'exception de M. Picard*," *Comptes Rendus*, Jan. 18, 1904.

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point of view, he has considered more general cases than I have, and so naturally obtained less precise results.

Mr. Barnes and other writers (among whom I may mention MM. Lindelöf and Wiman) have proved that $\Pi_\rho(x)$ is represented asymptotically by the expression

$$(2\pi)^{-\frac{1}{2}\rho} x^{-\frac{1}{2}} \exp. \left(\frac{\pi}{\sin \frac{\pi}{\rho}} x^{\frac{1}{\rho}} \right),$$

provided a certain region including the zeroes is excluded from consideration. My first object has been to find an expression which represents $P_\rho(x)$ asymptotically within this region, and especially for real negative values of n. I find that such an expression is given by

$$(2\pi)^{-\frac{1}{2}\rho} (-x)^{-\frac{1}{2}} 2 \sin \left\{ \pi (-x)^{\frac{1}{\rho}} \right\} \exp. \left\{ \pi \cot \frac{\pi}{\rho} (-x)^{\frac{1}{\rho}} \right\}.$$

This expression must be taken real for negative real values of x, and applies at any rate within an angle which has its vertex at the origin and includes the negative real axis.

Using these two asymptotic expressions I find that, if $1 < \rho < 2$, the large roots of the equation

$$P_\rho(x) = a_n x^n + \ldots + a_0$$

form two series lying in the directions

$$\theta = \pm \tfrac{1}{2}\rho\pi,$$

one set being given by the approximate formula

$$\left(\frac{\sin \frac{\pi}{\rho}}{\pi} \right)^\rho e^{\frac{1}{2}\rho\pi i} \{ (2p\pi)^\rho - i(n + \tfrac{1}{2}) \rho^2 (2p\pi)^{\rho-1} \log (2p) + \ldots \},$$

and the other by the conjugate formula, p being an integer which takes all large positive values. If $2 < \rho$, however, the roots approximate to the points $-p^\rho$. The case of $\rho = 2$ is of particular interest. For $\rho = 1$ the product Π_ρ becomes divergent. If we make it convergent by introducing the proper exponential factors we are led to the inverse Gamma-function. This case I have considered in a paper which will appear shortly in the *Proceedings of the London Mathematical Society.*

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It will be observed that, if N is the number of roots within a circle of radius r, we have roughly $N = r^{\frac{1}{\rho}} \operatorname{cosec} \frac{\pi}{\rho}$, whereas the same number for $\Pi_\rho(x)$ itself is $N = r^{\frac{1}{\rho}}$. *The ratio of these is* $\sin \frac{\pi}{\rho}$. This agrees with a remark made by Mr. Barnes and M. Wiman.

§ 15. It is evident that the equation $f(x) = 0$ will present similar peculiarities in large classes of cases; whenever for instance the zeroes of $f(x)$ are all real and negative, and the inequality

$$|f(x)| < e^{-\delta r^\alpha} \quad (\delta, \alpha > 0)$$

is satisfied throughout an angle which includes the line of zeroes. Thus it can be shown that the function

$$f(x) = \Pi\left(1 + \frac{x}{a_n}\right) \quad (Hn^\rho < a_n < Kn^\rho),$$

H, K being positive and $1 < \rho < 2$, is such a function. Moreover, we can easily construct cases in which the eccentricity of the equation is more complicated; we might consider, for instance, the equation

$$e^{ax^2} \sin bx = c.$$

To the equations (10) and (11), and the assertions which I have made about their roots, I hope to return on some future occasion.

I hope also to consider certain functions, defined by Taylor series, whose zeroes can be determined asymptotically by transforming the series into an integral of the type

$$\int_0^1 e^{xu} \phi(u)\, du.$$

The function

$$\frac{1}{p} \int_0^1 e^{xu} u^{\frac{1}{p}-1}\, du = 1 + \frac{x}{(p+1)\,1!} + \frac{x^2}{(2p+1)\,2!} + \ldots$$

is such a function. It is a generalisation of the exponential function and possesses very interesting properties.

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ON THE NEWTONIAN POTENTIAL.

By Prof. A. C. Dixon, Sc.D.

THE object of this paper is to discuss the question whether there is a theory bearing the same relation to the Newtonian potential, as that of the functions of a complex variable bears to the logarithmic potential. The functions discussed are variable quaternions or vectors destroyed by the operator ∇. The theory is made comparatively narrow by the fact that such functions cannot as a rule be combined, except by addition, without losing the characteristic property. Another restriction is that infinity cannot be treated as an ordinary point, for, while ∇^2 is not seriously affected by inversion, ∇ is quite altered. Much of the discussion relates to certain functions with triple periodicity, whose properties, so far as they go, are strikingly like those of elliptic functions.

§ 1. Let W, X be any two potentials, that is, functions of x, y, z, such that $\nabla^2 W = 0$, $\nabla^2 X = 0$, where ∇ is Hamilton's operator

$$i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}.$$

Let two functions Y, Z be taken, such that

$$\frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y} - \frac{\partial W}{\partial z}, \quad \frac{\partial Z}{\partial x} = \frac{\partial X}{\partial z} + \frac{\partial W}{\partial y};$$

this is just a matter of integration with respect to x, the other variables y, z being treated as constant.

It follows at once by differentiation that

$$\frac{\partial^2 Y}{\partial x \partial y} + \frac{\partial^2 Z}{\partial x \partial z} = \frac{\partial^2 X}{\partial y^2} + \frac{\partial^2 X}{\partial z^2} = -\frac{\partial^2 X}{\partial x^2},$$

and that $$\frac{\partial^2 Y}{\partial x \partial z} - \frac{\partial^2 Z}{\partial x \partial y} = -\frac{\partial^2 W}{\partial z^2} - \frac{\partial^2 W}{\partial y^2} = \frac{\partial^2 W}{\partial x^2};$$

thus $$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \text{ and } \frac{\partial W}{\partial x} - \frac{\partial Y}{\partial z} + \frac{\partial Z}{\partial y}$$

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are functions of y, z only. Let

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=f(y, z),$$

$$\frac{\partial W}{\partial x}-\frac{\partial Y}{\partial z}+\frac{\partial Z}{\partial y}=\phi(y, z).$$

Now η, ζ, functions of y, z, can be so determined that

$$\frac{\partial \eta}{\partial y}+\frac{\partial \zeta}{\partial z}=f(y, z),$$

$$-\frac{\partial \eta}{\partial z}+\frac{\partial \zeta}{\partial y}=\phi(y, z).$$

For, if we put

$$y+\iota z=u,\quad y-\iota z=v,\quad \eta+\iota\zeta=U,\quad \eta-\iota\zeta=V,$$

and take u, v as independent variables, these conditions may be written

$$2\frac{\partial U}{\partial u}=f\left(\frac{u+v}{2},\ \frac{u-v}{2\iota}\right)+\iota\phi\left(\frac{u+v}{2},\ \frac{u-v}{2\iota}\right),$$

$$2\frac{\partial V}{\partial v}=f\left(\frac{u+v}{2},\ \frac{u-v}{2\iota}\right)-\iota\phi\left(\frac{u+v}{2},\ \frac{u-v}{2\iota}\right),$$

and the solution again is only a matter of integration.

If then we take $Y-\eta$, $Z-\zeta$ in the place of Y, Z, we have four functions W, X, Y, Z, such that

$$\frac{\partial W}{\partial x}-\frac{\partial Y}{\partial z}+\frac{\partial Z}{\partial y}=0,$$

$$\frac{\partial W}{\partial y}+\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}=0,$$

$$\frac{\partial W}{\partial z}-\frac{\partial X}{\partial y}+\frac{\partial Y}{\partial x}=0,$$

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0,$$

that is, such that

$$\nabla(W+iX+jY+kZ)=0.$$

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We shall call a quaternion such as $W+iX+jY+kZ$, which is destroyed by ∇, a Hamiltonian function; it appears to be the analogue of a function of a complex variable in the plane.

The four functions W, X, Y, Z must all be potentials, since

$$\nabla^2(W+iX+jY+kZ) = \nabla 0 = 0.$$

§ 2. Hamiltonian functions may be constructed in another way, as follows. Let P, Q, R, S be *any* four potentials, then, since $\nabla^2(P+iQ+jR+kS)=0$, it follows that

$$\nabla(P+iQ+jR+kS)$$

is a Hamiltonian function. The theory of these functions will thus be little else than an arrangement of what is known about the Newtonian potential.

The product of two Hamiltonian functions will not generally be another; even when a function is multiplied by a constant, the constant must be the second factor or the Hamiltonian character is lost unless the constant is a pure scalar.

§ 3. There is a process corresponding to that of integration round a closed curve, depending on the theorem that $\int(d\sigma . U)=0$, where U is a Hamiltonian function without singularity within or upon the closed surface over which the integration is taken, and $d\sigma$ is an element of the surface treated as a vector, that is, the area of the element multiplied by the unit vector in the direction of the outward normal. The proof of this depends on the ordinary formula for the change of a surface integral into a volume integral, for we have at once

$$\begin{aligned}\int(d\sigma . U) &= \iiint\left(i\frac{\partial U}{\partial x}+j\frac{\partial U}{\partial y}+k\frac{\partial U}{\partial z}\right)dx\,dy\,dz \\ &= \iiint \nabla U . dx\,dy\,dz \\ &= 0,\end{aligned}$$

since U is Hamiltonian.

In the same way $\int KU . d\sigma = 0$, KU being the quaternion conjugate to U, for we have

$$\frac{\partial}{\partial x}KU.i+\frac{\partial}{\partial y}KU.j+\frac{\partial}{\partial z}KU.k=0.$$

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Again, if P is any scalar potential, we have, for any closed surface within and on which U, P have no singularity,

$$\int(KU.d\sigma.\nabla P)=\iiint\left[\frac{\partial}{\partial x}(KU.i\nabla P)\right.$$
$$\left.+\frac{\partial}{\partial y}(KU.j\nabla P)+\frac{\partial}{\partial z}(KU.k\nabla P)\right]dx\,dy\,dz$$
$$=\iiint(KU.\nabla^2 P)\,dx\,dy\,dz$$
$$+\iiint\left[\frac{\partial}{\partial x}KU.i+\frac{\partial}{\partial y}KU.j+\frac{\partial}{\partial z}KU.k\right]\nabla P.dx\,dy\,dz$$

$=0$. Similarly $\int\nabla P.d\sigma.U=0$.

§ 4. In these results we may put $P=\frac{1}{r}=(x^2+y^2+z^2)^{-\frac{1}{2}}$, and thus $\int\left(KU.d\sigma.\nabla\frac{1}{r}\right)$, for instance, has the same value when taken over any closed surface enclosing the origin, as it has when taken over a small sphere with the origin as centre. But on such a sphere

$$d\sigma=(ix+jy+kz)\frac{dS}{r},$$

dS being the area of the element,

$$\nabla\frac{1}{r}=-\frac{1}{r^3}(ix+jy+kz);$$

thus $d\sigma.\nabla\frac{1}{r}$ is scalar, and its value is in fact the solid angle subtended at the origin by the element of area. Hence $\frac{1}{4\pi}\int\left(KU.d\sigma.\nabla\frac{1}{r}\right)$ is the mean value of KU on the spherical surface, which must coincide with its value at the centre since the sphere may be as small as we please.

It follows that if $\rho=\{(x-x')^2+(y-y')^2+(z-z')^2\}^{\frac{1}{2}}$, U' the value of U at (x', y', z') is equal to

$$\frac{1}{4\pi}K\int\left(KU.d\sigma.\nabla\frac{1}{\rho}\right)\text{ or to }\frac{1}{4\pi}\int\left(\nabla\frac{1}{\rho}.d\sigma.U\right),$$

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the integral being taken over any closed surface enclosing (x', y', z'), but not any singularity of U. Since $\nabla \frac{1}{\rho}$ is a function of $x-x', y-y', z-z'$ only, we have

$$\frac{1}{4\pi} K \int KU.d\sigma . \frac{\partial^{a+b+c}}{\partial x^a \partial y^b \partial z^c} \nabla \frac{1}{\rho}$$

$$= (-1)^{a+b+c} \frac{\partial^{a+b+c} U'}{\partial x'^a \partial y'^b \partial z'^c} = \frac{1}{4\pi} \int \frac{\partial^{a+b+c}}{\partial x^a \partial y^b \partial z^c} \nabla \frac{1}{\rho} . d\sigma . U.$$

By expanding $\frac{1}{\rho}$ in ascending powers and products of x', y', z' we have an expansion for U' analogous to Maclaurin's theorem, holding good within a sphere with the origin as centre; there will in the same way be a double expansion analogous to Laurent's, since $\frac{1}{\rho}$ may be expanded also in ascending powers and products of x, y, z. This will hold within the region bounded by two concentric spheres.

§ 5. Of the rational integral algebraic Hamiltonian functions of degree n, $n+1$ are to be counted distinct. The most general expression of degree n contains $\frac{1}{2}(n+1)(n+2)$ quaternion coefficients or $2(n+1)(n+2)$ numerical coefficients. If U is of the degree n, then ∇U is of the degree $n-1$ and contains $2n(n+1)$ numerical coefficients. Each of these must vanish, and the number of numerical coefficients in U, if it is Hamiltonian, is thus $4(n+1)$. This must be divided by 4, since U may be post-multiplied by an arbitrary quaternion constant.

For the degree 1 we have the functions $x - ky$, $x + jz$;

For the degree 2 we have the functions

$$x^2 - 2kxy - y^2, \; x^2 + 2jxz - z^2, \; y^2 - 2iyz - z^2;$$

and so on.

The expressions $x - ky$, $z - jx$, $y - iz$ are ordinary complex variables and the functions derived from each in the ordinary complex theory will be Hamiltonian functions: the same holds for monogenic functions of such a complex variable as

$$(\lambda_1 x + \mu_1 y + \nu_1 z) + (i\lambda_2 + j\mu_2 + k\nu_2)(\lambda_3 x + \mu_3 y + \nu_3 z),$$

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where $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, $(\lambda_3, \mu_3, \nu_3)$ are the direction-cosines of three mutually perpendicular straight lines so arranged that

$$\begin{vmatrix} \lambda_1, & \mu_1, & \nu_1 \\ \lambda_2, & \mu_2, & \nu_2 \\ \lambda_3, & \mu_3, & \nu_3 \end{vmatrix} = 1.$$

Among functions of negative degrees we have for the degree -2 the function

$$(ix + jy + kz)/r^3,$$

for the degree -3 the functions

$$\{i(y^2 + z^2 - 2x^2) - 3jxy - 3kxz\}/r^5,$$

$$\{j(z^2 + x^2 - 2y^2) - 3ixy - 3kyz\}/r^5,$$

and so on.

Let $\phi(-n)$ be the number of such functions for the degree $-n$; these arise from operating with ∇ upon Y_{n-2}/r^{2n-3}, where Y_{n-2} is the general surface harmonic of degree $n-2$ and includes $2n-3$ constants, quaternions, or $4(2n-3)$ numerical constants. But there are $4\phi(-n+1)$ expressions, included in Y_{n-2}/r^{2n-3}, which ∇ destroys; hence

$$4\phi(-n) = 4(2n-3) - 4\phi(-n+1),$$

and

$$\phi(-n) = n - 1, \text{ since } \phi(-2) = 1.$$

These are the functions of negative degree that will enter into the expansions referred to above analogous to Laurent's. There are others, such as for instance the negative powers of $x - ky$.

§ 6. Any Hamiltonian function must become infinite or otherwise singular at some point either at a finite or an infinite distance, for a potential cannot have a true maximum or minimum in empty space.

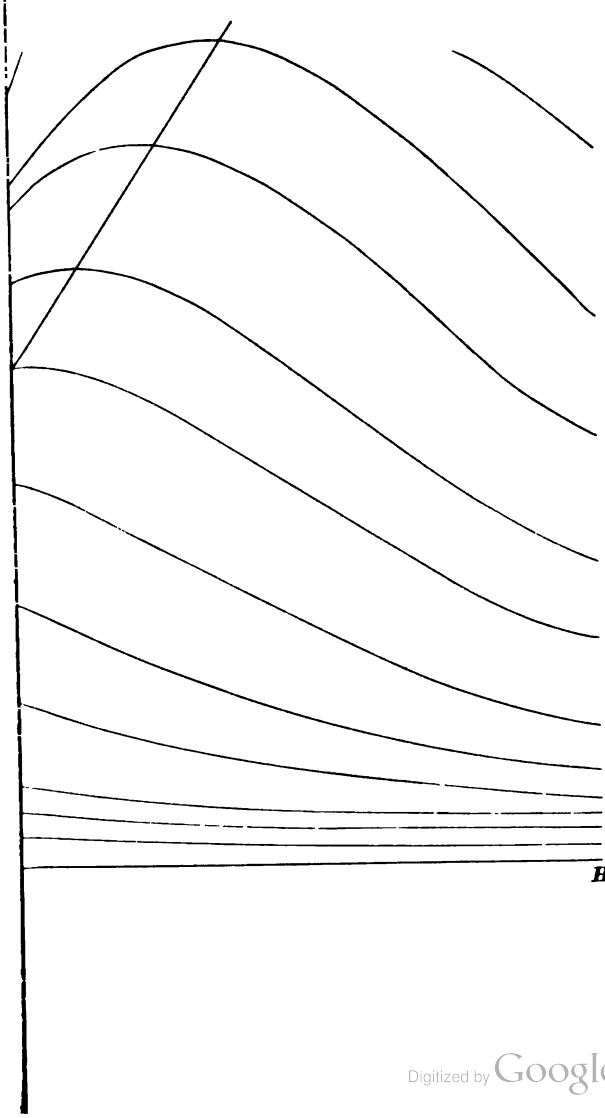
When a one-valued Hamiltonian function becomes singular at an isolated point, the Laurent expansion shows that its infinite part is an aggregate of multiples of such expressions as

$$\nabla \frac{\partial^{a+b+c}}{\partial x^a \partial y^b \partial z^c} \frac{1}{\rho}.$$

This does not hold if the function becomes infinite along a line, or at least in such a case the aggregate does not consist of a finite number of terms.

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Fig. 1.



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WITH PLATE 1.

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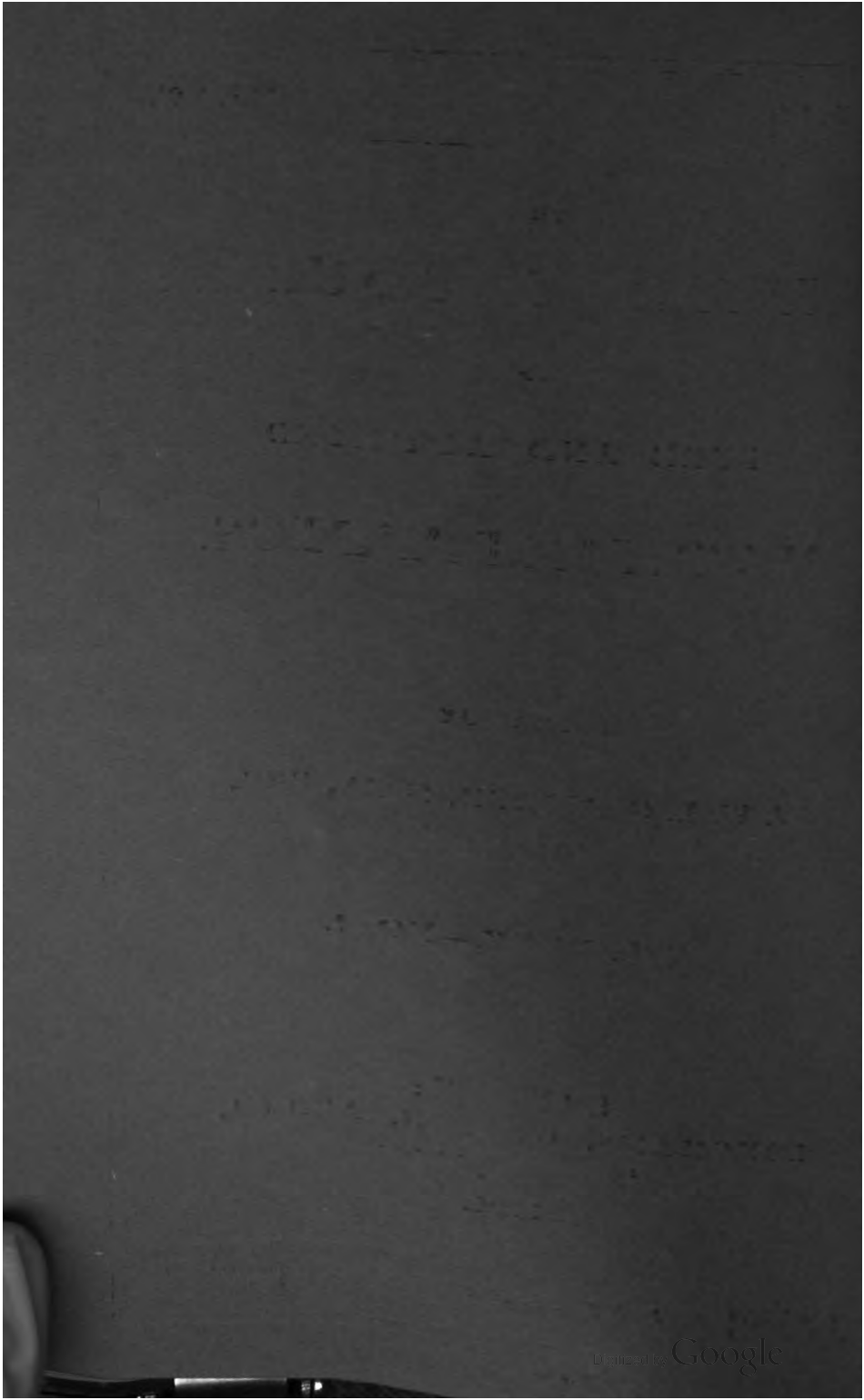
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§ 7. There are some special features about Hamiltonian functions that have no scalar part; let us call them pure functions. It is readily verified that if V is a pure function and α any constant quaternion then $K(\alpha V)$ is Hamiltonian. The elements of the surface integral $\frac{1}{4\pi} K \int \left(KU.d\sigma.\nabla \frac{1}{\rho} \right)$ are in fact of this type.

The values over a closed surface of $\int (Vd\sigma)$ and of $\int (d\sigma.V)$ both vanish if V is a pure function without singularity within or on the surface.

If $iX+jY+kZ$ is a pure function, we have

$$\nabla (iX+jY+kZ)=0,$$

or

$$\frac{\partial Y}{\partial z}=\frac{\partial Z}{\partial y}, \quad \frac{\partial Z}{\partial x}=\frac{\partial X}{\partial z}, \quad \frac{\partial X}{\partial y}=\frac{\partial Y}{\partial x},$$

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0,$$

thus X, Y, Z are the derivatives of a potential P, and any pure function may be put in the form ∇P.

§ 8. It is easy to construct triply-periodic Hamiltonian functions with arbitrary periods. Let the proposed periods be ω_1, ω_2, ω_3, where

$$\omega = i\alpha + j\beta + k\gamma,$$

Then

$$\sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \sum_{m_3=-\infty}^{\infty} \frac{\partial^s}{\partial x^s} \nabla \{ (x - m_1\alpha_1 - m_2\alpha_2 - m_3\alpha_3)^2 + (y - m_1\beta_1 - m_2\beta_2 - m_3\beta_3)^2 + (z - m_1\gamma_1 - m_2\gamma_2 - m_3\gamma_3)^2 \}^{-\frac{1}{2}}$$

is such a function; in the place of $\frac{\partial^s}{\partial x^s}$ we may use any other differential operator of at least the second order, formed with $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$. This restriction on the order secures convergency.

Let

$$\nabla \{ (x - m_1\alpha_1 - m_2\alpha_2 - m_3\alpha_3)^2 + (y - m_1\beta_1 - m_2\beta_2 - m_3\beta_3)^2 + (z - m_1\gamma_1 - m_2\gamma_2 - m_3\gamma_3)^2 \}^{-\frac{1}{2}}$$

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be denoted by $f(x, y, z, m_1, m_2, m_3)$; then

$$\sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \sum_{m_3=-\infty}^{\infty} [f(x-p, y-q, z-r, m_1, m_2, m_3) - f(x_0-p, y_0-q, z_0-r, m_1, m_2, m_3) - f(x-p_0, y-q_0, z-r_0, m_1, m_2, m_3) + f(x_0-p_0, y_0-q_0, z_0-r_0, m_1, m_2, m_3)]$$

is another triply periodic pure function if

$$p, q, r, x_0, y_0, z_0, p_0, q_0, r_0$$

are any finite quantities. Let this function be called $F(\xi, \xi_0; \varpi, \varpi_0)$, where $\xi = ix + jy + kz$, $\varpi = ip + jq + kr$, etc. Then F has the value zero when $\xi = \xi_0$ and has isolated infinities of the simplest possible kind at ϖ, ϖ_0: we may call ϖ, ϖ_0 the poles of F. Any triply periodic one-valued Hamiltonian function must have at least two poles of this kind or one of a higher order. For let $\phi(\xi)$ be any triply periodic Hamiltonian function, having no singularity within the period-parallelepiped but a simple pole at η, (u, v, w), and behaving near η like the function $\nabla \frac{1}{|\xi - \eta|} . \alpha$, where α is a constant quaternion and $|\xi - \eta|$ denotes the distance between the points ξ, η. Then zero is the value of

$$\int F(\xi, \xi_0; \varpi, \varpi_0) . d\sigma . \phi(\xi)$$

taken with respect to ξ over the surface of the period-parallelepiped, since the contributions of corresponding parts of opposite faces destroy each other. But F is a pure function and the only singularities of the subject of intergration occur at η, ϖ, ϖ_0, while the values of the integral over small spheres containing these points are

$$4\pi F(\eta, \xi_0; \varpi, \varpi_0)\alpha;\ 4\pi\phi(\varpi);\ -4\pi\phi(\varpi_0).$$

Hence by the results of § 3,

$$4\pi\phi(\varpi) - 4\pi\phi(\varpi_0) + 4\pi F(\eta, \xi_0; \varpi, \varpi_0).\alpha = 0,$$

and here ϖ, ϖ_0, ξ_0 are arbitrary. Putting $\xi_0 = \eta$ we have $\phi(\varpi) = \phi(\varpi_0)$, so that ϕ has the same value at two arbitrary points ϖ, ϖ_0 and is therefore constant; α must be zero.

But if we take ϕ to have two poles, η, η_0, and use the same notation and method we have

$$\phi(\varpi) - \phi(\varpi_0) = -F(\eta, \xi_0; \varpi, \varpi_0)\alpha - F(\eta_0, \xi_0; \varpi, \varpi_0)\alpha_0.$$

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Putting η, η_0 for ξ_0 in turn, we find $\alpha = -\alpha_0$ and

$$\phi(\varpi) = \phi(\varpi_0) - F(\eta, \eta_0;\ \varpi, \varpi_0)\,\alpha$$
$$= \phi(\varpi_0) + F(\varpi, \varpi_0;\ \eta, \eta_0)\,\alpha.$$

Since ϖ is arbitrary this shews that $\phi(\xi)$ only differs by a constant from a post-multiple of $F(\xi, \xi_0;\ \eta, \eta_0)$.

Now let ϕ have a pole of a higher order at η, and in fact assume that ϕ becomes infinite like

$$\frac{\partial^{a+b+c}}{\partial x^a \partial y^b \partial z^c} \nabla \frac{1}{|\xi - \eta|}\,\alpha,$$

and has no other pole in the parallelepiped. Since

$$\eta = iu + jv + kw,$$

the same method gives

$$\phi(\varpi) = \phi(\varpi_0) + (-1)^{a+b+c} \frac{\partial^{a+b+c}}{\partial u^a \partial v^b \partial w^c} F(\varpi, \varpi_0;\ \eta, \eta_0).\alpha.$$

Thus the function F and its derivatives enable us to express all possible triply-periodic Hamiltonian functions whose distinct poles are finite in number and of finite order.

§ 9. Some of the properties of the function F are suggested by those of doubly periodic functions. For instance,

$$F(\varpi + \varpi_0 - \xi, \xi_0;\ \varpi, \varpi_0) = F(\xi, \xi_0;\ \varpi, \varpi_0);$$

or distinct places at which F has the same value occur in pairs whose mean is fixed.

Taking any vector $iA + jB + kC$, we may prove as follows that it is the value of F at some point in the parallelepiped. Since $\nabla F = 0$ and F is a vector we may put $F = \nabla P$ where P is a scalar potential. The singularities of P in the period block are those of $\frac{1}{|\xi - \varpi|}$ and $-\frac{1}{|\xi - \varpi_0|}$. Consider the surfaces upon which $P - Ax - By - Cz$ is constant.

For great positive values these approximate to spheres round the point ϖ and those corresponding to it, and for great negative values to spheres round the point ϖ_0 and those corresponding to it.

In the transition from one of these forms to the other the surface must pass through forms in which it has a node, and at such a node the derivatives of $P - Ax - By - Cz$ must vanish and

$$F = iA + jB + kC.$$

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§ 10. The further theorem, that there are only two points in the period block at which F takes a particular value, if true at all, is subject to exception.

For, take the periods to be

$$jb + kc,\ kc + ia,\ ia + jb$$

where a, b, c are scalar; form the function

$$F(\xi, 0;\ \tfrac{1}{2}ia + \tfrac{1}{2}jb + \tfrac{1}{2}kc,\ -\tfrac{1}{2}ia + \tfrac{1}{2}jb + \tfrac{1}{2}kc).$$

The coefficients of j, k in this vanish if $x = 0$, and thus the plane $x = 0$ contains an infinite series of curves along each of which F is a constant multiple of i. The same holds in the planes $y = 0$, $z = 0$. The function F in this case expresses the force due to an electrified particle at the centre of a rectangular hole in a conductor, whose edges are a, b, c. This force at any point in the surface of the hole is normal to the surface.

§ 11. The analogue of the elliptic function of the second kind is

$$-\int_{\xi_0}^{\xi}\left(\frac{\partial F}{\partial p}dx + \frac{\partial F}{\partial q}dy + \frac{\partial F}{\partial r}dz\right),$$

or $\Sigma\Sigma\Sigma\,[f(x-p, y-q, z-r, m_1, m_2, m_3)$

$$-f(x_0-p, y_0-q, z_0-r, m_1, m_2, m_3)$$

$$-\left\{(x-x_0)\frac{\partial}{\partial x_0} + (y-y_0)\frac{\partial}{\partial y_0}\right.$$

$$\left. + (z-z_0)\frac{\partial}{\partial z_0}\right\} f(x_0-p, y_0-q, z_0-r, m_1, m_2, m_3)].$$

This is a function whose only singularity in the parallelepiped is a pole at ϖ, and it is increased by a constant quantity when ξ is increased by a period.

Let $Z(\xi, \xi_0)$ denote the result of putting $\varpi = 0$ in the function. Then we have for $F(\xi, \xi_0;\ \varpi, \varpi_0)$ the expression

$$Z(\xi - \varpi, \eta) - Z(\xi_0 - \varpi, \eta) - Z(\xi - \varpi_0, \eta) + Z(\xi_0 - \varpi_0, \eta),$$

where η is an arbitrary vector.

The constants by which $Z(\xi, \xi_0)$ is increased when ξ is increased by a period satisfy a certain relation.

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Let $\phi(\xi)$ be the most general function having the same properties as Z, that is, having no singularity but that of $\nabla \frac{1}{|\xi - \varpi|}$ in the parallelepiped and such that

$$\phi(\xi + \omega_n) - \phi(\xi) = \vartheta_n \ (n = 1, 2, 3),$$

where ϑ_1, ϑ_2, ϑ_3 are constant vectors. Then

$$\int \phi(\xi)\, d\sigma$$

taken over the surface of the parallelepiped is 4π, but by taking opposite faces together we find for it the value

$$\vartheta_1\Omega_1 + \vartheta_2\Omega_2 + \vartheta_3\Omega_3,$$

where Ω_1, Ω_2, Ω_3 are the vector values of the areas of the faces. Hence

$$\vartheta_1\Omega_1 + \vartheta_2\Omega_2 + \vartheta_3\Omega_3 = 4\pi.$$

This quaternion equation is equivalent to four conditions, and thus the three vectors ϑ_1, ϑ_2, ϑ_3 can only depend on five arbitrary scalar constants.

These five are accounted for by the fact that

$$\phi(\xi) + \alpha x + \beta y + \gamma z,$$

where α, β, γ are constant vectors such that

$$i\alpha + j\beta + k\gamma = 0,$$

is a pure Hamiltonian function having the same properties as were assumed for $\phi(\xi)$, but with different values of ϑ_1, ϑ_2, ϑ_3. There are five scalar arbitrary constants left in α, β, γ when the above condition is satisfied.

Hence the function $\phi(\xi)$ can be none other than

$$Z(\xi - \varpi, \xi_0)$$

+ an expression linear in x, y, z; in fact α, β, γ can be so determined that

$$i\alpha + j\beta + k\gamma = 0,$$

and that

$$\phi(\xi) + \alpha x + \beta y + \gamma z - Z(\xi - \varpi, \xi_0)$$

is triply-periodic; then since it has no singularity it must be constant.

§ 12. Since F is a Hamiltonian function we have $\int d\sigma . F = 0$ when the surface of integration is closed and does not surround any of the points corresponding to the poles ϖ, ϖ_0. Hence $\int d\sigma . F$ has the same value over two surfaces

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whose boundary is the same provided that no pole lies in the space between them. This, the surface integral of a triply periodic function, is the analogue of the third elliptic integral. It has two parameter points, ϖ, ϖ_0, but the place of the argument and zero points is taken by the curve which bounds the surface of integration. A change in ξ_0 simply alters the subject of integration by adding to it a constant vector.

The double integral is a triply periodic function of ϖ or ϖ_0, since F is. If the surface of integration is deformed so as to pass over ϖ or ϖ_0 the double integral is increased by $\pm 4\pi$, the sign depending on the direction in which the passage takes place, and on whether the pole passed over is ϖ or ϖ_0.

If the surface of integration undergoes a simple translation represented by one of the vectors ω_1, ω_2, ω_3, the value of the integral is unaffected. Hence the integral vanishes when taken over a tubular surface bounded by two curves corresponding to each other in different parallelepipeds, if no pole is enclosed when the ends of the tube are stopped by corresponding surfaces: the same would be true also if the number of points enclosed corresponding to ϖ were equal to the number corresponding to ϖ_0.

§ 13. When the boundary of the surface passes near the point ϖ or ϖ_0 there is a singularity like that of $\pm\int d\sigma . \nabla \frac{1}{r}$ when the boundary passes near the origin.

Now the scalar part of $\int d\sigma . \nabla \frac{1}{r}$ is the solid angle subtended at the origin by the surface of integration.

The vector part is

$$\int \frac{1}{r^3} [i\{yd(x, y) + zd(x, z)\} + j\{xd(y, x) + zd(y, z)\} + k\{xd(z, x) + yd(z, y)\}],$$

where $d(x, y)$, for instance, stands for $dxdy$ and it is to be remembered that

$$d(x, y) = -d(y, x),$$
$$d(x, x) = 0.$$

The expression may be written more shortly

$$\int d\left(\frac{1}{r}, ix + jy + kz\right) \text{ or } \int d\left(\frac{1}{|\xi|}, \xi\right).$$

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It vanishes when the boundary of the surface of integration lies on a sphere about the origin as centre, since on such a sphere $|\xi|$ is constant: it corresponds to the logarithmic term in two dimensions, which also vanishes for $\int du/u$ when the limiting values of the complex variable u lie on a circle with centre 0.

§ 14. The periodic constants for the double integral are got by taking for the surface of integration a face of the period parallelepiped. Let the three constants be E_1, E_2, E_3 arising from the faces $\omega_2\omega_3$, $\omega_3\omega_1$, $\omega_1\omega_2$ respectively.

E_1, E_2, E_3 are quaternions involving twelve scalar constants. Relations connecting these constants may be found as follows. Since $jx+iy$ is a Hamiltonian function the value of

$$\int (jx+iy)\, d\sigma . F,$$

taken over the surface of the parallelepiped, is $4\pi\times$ the difference of the values of $jx+iy$ at ϖ, ϖ_0 the poles of F, that is $4\pi j(p-p_0)+4\pi i(q-q_0)$.

But the contribution of the two faces $\omega_2\omega_3$ is

$$(j\alpha_1+i\beta_1)\int_{\omega_2\omega_3} d\sigma . F, \text{ or } (j\alpha_1+i\beta_1)\, E_1.$$

Similarly for the other pairs of faces; thus

$$j\Sigma\alpha E+i\Sigma\beta E=4\pi j(p-p_0)+4\pi i(q-q_0).$$

Another relation of this kind is given by taking $ky+jz$ in place of $jx+iy$, and from the two it follows that

$$\Sigma\alpha E=4\pi(p-p_0)+iH,$$

$$\Sigma\beta E=4\pi(q-q_0)+jH,$$

$$\Sigma\gamma E=4\pi(r-r_0)+kH,$$

where H is some quaternion.

We may reduce H to zero by adding to F a properly chosen quaternion constant. For if $F+h$ is put in place of F, the values of E_1, E_2, E_3 become

$$E_1+\Omega_1 h,\ E_2+\Omega_2 h,\ E_3+\Omega_3 h.$$

Now $\Sigma\alpha\Omega=i\begin{vmatrix}\alpha_1, & \beta_1, & \gamma_1\\ \alpha_2, & \beta_2, & \gamma_2\\ \alpha_3, & \beta_3, & \gamma_3\end{vmatrix}=iM$, say,

$$\Sigma\beta\Omega=jM,\ \Sigma\gamma\Omega=kM,$$

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and thus the new value of $\Sigma\alpha E$ is

$$4\pi(p-p_0)+iH+iMh.$$

Thus if we put $F-H/M$ in place of F we have

$$\Sigma\alpha E = 4\pi(p-p_0),$$

$$\Sigma\beta E = 4\pi(q-q_0),$$

$$\Sigma\gamma E = 4\pi(r-r_0).$$

The three periodic constants E_1, E_2, E_3 are therefore now scalar, and depend linearly on $\varpi-\varpi_0$. From the fact that E_1, E_2, E_3 are integrals over plane areas, it may be proved without difficulty that the modification in F does not affect its scalar part, so that it is still a pure function.

The surface of integration that gives the value E_1 may be deformed very freely and its boundary is by no means fixed. It may be any surface bounded by four lines, of which the third is what the first would become after a translation ω_2, and the fourth is what the second would become after a translation ω_3. Similarly for E_2, E_3. The periodic constants will be altered by $\pm 4\pi$ if the surfaces are deformed over ϖ or ϖ_0, or a point corresponding.

§ 15. The elliptic theta function is that whose logarithm is the integral of the elliptic function of the second kind. The surface integral of $Z(\xi, \xi_0)$ will therefore take the place of the logarithm of the theta function and will have analogous properties. I do not think it worth while to discuss these further, since there seems to be nothing in this direction to correspond with the theta function itself.

An extension to functions of four variables suggests itself, these being quaternions, functions of w, x, y, z, and destroyed by the operator

$$\frac{\partial}{\partial w}+i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}.$$

Functions of the two complex variables $w+ix$, $z+iy$ would be included.

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ON FOCI AND CONFOCAL PLANE CURVES.

By R. A. Roberts.

I. *On some curves with their foci connected in a certain manner.*

THE manner in which the points are connected is simply that the pencils joining the points to I, J, the circular points, are both self-apolar (or equianharmonic) tetrads. Hence, if a plane curve be described to have the four points as foci, the entire sixteen foci are the intersections of the two tetrads. From the equivalence of the anharmonic ratios the sixteen points lie by fours on four circles (Salmon's *Higher Plane Curves*, Art. 168), but four real points cannot lie on one circle. Let A, B be two real foci, then the two other real foci C, D are the antipoints of two points C', D' on a circle through AB, and such that A, B, C', D' are a self-apolar tetrad of points. Hence, if A, B, C, D are the real foci, we have

$$AD.BC = BD.CA = CD.AB = \sqrt{\left(\frac{2a^2b^2c^2}{(a^2+b^2+c^2 \pm 4\Delta\sqrt{3})}\right)} \quad \ldots\ldots\ldots(1),$$

where a, b, c, Δ are the sides and area, respectively, of the triangle A, B, C. Taking trilinear coordinates with regard to the triangle ABC we see hence that the point D is the inverse point (inverse points are such that $\alpha\alpha' = \beta\beta' = \gamma\gamma'$) of one of the two points at which the three sides subtend angles of 120°; thus, for D,

$$\alpha : \beta : \gamma = \cos A \pm \sqrt{3}\sin A : \cos B \pm \sqrt{3}\sin B : \cos C \pm \sqrt{3}\sin C \quad \ldots\ldots\ldots(2).$$

Or, D is the point which inverts (in the ordinary sense) ABC into an equilateral triangle.

Now considering a bicircular quartic with foci connected in this manner, we have the peculiarity that there are three distinct confocal systems instead of one as in the ordinary case. I observe that if a binary quartic is capable of being written in the form

$$(x-a)\{m^3(x-a)^3 + n^3(x-a')^3\},$$

then its roots are self-apolarly connected. Now, taking the differential equation

$$\frac{du}{\sqrt{(1+u^3)}} \pm \frac{dv}{\sqrt{(1+v^3)}} = 0 \quad \ldots\ldots\ldots(3),$$

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we know that the integral gives a rational (2, 2) relation between u, v. Putting then

$$u = \frac{n(x-a')}{m(x-a)}, \quad v = \frac{n'(y-b')}{m'(y-b)} \quad \ldots\ldots\ldots\ldots(4),$$

(3) becomes $$\frac{mn(a-a')\,dx}{\sqrt{X}} \pm \frac{m'n'(b-b')\,dy}{\sqrt{Y}} = 0 \quad \ldots\ldots(5),$$

where $$m(x-a)\{m^2(x-a)^2 + n^2(x-a')^2\} = X,$$
$$m'(y-b)\{m'^2(y-b)^2 + n'^2(y-b')^2\} = Y,$$

and x, y are circular coordinates,. Hence, as Darboux has shown, we have the differential equation of the mutually orthogonal systems of confocal bicircular quartics. But it is to be observed that we have still an integral of (3) of the same form, if we substitute θv or $\theta^2 v$ for v, where θ is an imaginary cube root of unity.

We thus have in this case three distinct systems of confocal bicircular quartics given by the differential equations

$$\frac{dx}{\sqrt{X}} \pm \frac{\mu dy}{\sqrt{Y}} = 0, \quad \frac{dx}{\sqrt{X}} \pm \frac{\mu\theta dy}{\sqrt{Y}} = 0, \quad \frac{dx}{\sqrt{X}} \pm \frac{\mu\theta^2 dy}{\sqrt{Y}} = 0 \ldots(6),$$

where $$\frac{m'n'(b-b')}{mn(a-a')} = \mu.$$

Now, recollecting the meaning of $\frac{dy}{dx}$, namely $e^{i\theta}$, where θ is the angle the tangent of the curve makes with a fixed line, we see that a curve of one system meets a curve of another system at an angle of 120°, 60°, or 30°,

I now proceed to consider a nodal cubic with a self-apolar tetrad of foci in regard to the circular points. Using circular coordinates and expressing the curve parametrically I write

$$x : y : 1 = f_1 : f_2 : f_3 \quad \ldots\ldots\ldots\ldots(7).$$

Now the tangents parallel to x are found by taking the discriminant of $f_1 - xf_3$ with regard to the parameter λ. But by the theory of two binary cubics, if two cubics are apolar, the binary quartic obtained by taking the discriminant of $f_1 - xf_3$ is self-apolar. Taking the parameter now so that $f_3 = 1 + \lambda^3$, $f_1 = a\lambda^3 + 3b\lambda^2 + 3c\lambda + d$, and expressing that f_3, f_1 are apolar, we get $d = a$. Hence, transferring the origin to $x = a$, $y = a'$, we obtain the curve in the form

$$x = \frac{3b\lambda^2 + 3c\lambda}{1 + \lambda^3}, \quad y = \frac{3b'\lambda^2 + 3c'\lambda}{1 + \lambda^3} \quad \ldots\ldots(8),$$

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where it is evident that the origin is the node. From these equations the foci are found to be given by

$$\left.\begin{aligned} x^4 - 6bcx^2 - 4(b^2 + c^2)x - 3b^2c^2 &= 0 \\ y^4 - 6b'c'y^2 - 4(b'^2 + c'^2)y - 3b'^2c'^2 &= 0 \end{aligned}\right\} \quad \ldots\ldots(9).$$

From these equations we see that the centre of gravity of the foci is the node of all the determined curves. To find the curve when the foci are given in this case, I observe that from (8) we see that the three inflexions are given by $\lambda^3 + 1 = 0$, that is, the three points of inflexion are at infinity. Hence using a real coordinate it is easily seen that the equation of the curve is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(10),$$

or

$$(x + y + z)^3 - 27xyz = 0.$$

The tangential equation is then

$$\lambda^{-\frac{1}{2}} + \mu^{-\frac{1}{2}} + \nu^{-\frac{1}{2}} = 0 \quad \ldots\ldots\ldots\ldots\ldots(11).$$

Now let a_1b_1, a_2b_2, a_3b_3 be the circular coordinates of the vertices of the triangle formed by the inflexional tangents, then from (11) we see that the foci are given by

$$\left.\begin{aligned} (x - a_1)^{-\frac{1}{2}} + (x - a_2)^{-\frac{1}{2}} + (x - a_3)^{-\frac{1}{2}} &= 0 \\ (y - b_1)^{-\frac{1}{2}} + (y - b_2)^{-\frac{1}{2}} + (y - b_3)^{-\frac{1}{2}} &= 0 \end{aligned}\right\} \quad \ldots(12).$$

Hence, being given the foci, we determine six curves, for, the two cubic equations whose roots are a_1, a_2, a_3 and b_1, b_2, b_3 respectively, being given, we determine the vertices of the triangle formed by the inflexional tangents as follows: namely, if the result of clearing (13) of radicals is

$$x^4 + bcx^3 + 4dx + e = 0, \quad y^4 + bc'y^3 + 4d'y + e' = 0,$$

then the vertices of the triangles formed by the inflexional tangents are given by

$$x^3 + 3cx + d = 0, \quad y^3 + 3c'y + d' = 0;$$

that is, there are six triangles given by

a_1b_1, a_2b_2, a_3b_3; a_1b_1, a_2b_3, a_3b_2; a_1b_3, a_2b_2, a_3b_1;

a_1b_2, a_2b_1, a_3b_3; a_1b_2, a_2b_3, a_3b_1; a_1b_3, a_2b_1, a_3b_2...(13).

By inverting the preceding figures with regard to the node of the cubic we get results involving the quartic, with a triple point, passing through the circular points at infinity.

Inverting the equations (8) we get

$$x : y : 1 = (1 + \lambda^3)(b\lambda + c) : (1 + \lambda^3)(b'\lambda + c') : 3\lambda(b\lambda + c)(b'\lambda + c') \quad \ldots\ldots\ldots(14),$$

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which represents a quartic with a triple point at the origin and passing through the circular points. Since the foci of this curve from (9) are given by

$$\left.\begin{aligned} 3b'^2c'^2x^4 + 4(b'^3 + c'^3)x^3 + 6b'c'x^2 - 1 = 0 \\ 3b^2c^2y^4 + 4(b^3 + c^3)y^3 + 6bcy^2 - 1 = 0 \end{aligned}\right\} \quad \ldots(15),$$

it follows that the foci form a self-apolar tetrad. Again, since the coefficients of x and y are wanting in (15), we have $\Sigma \frac{1}{x_1} = 0$, $\Sigma \frac{1}{y_1} = 0$, that is, the origin, namely the triple point of the curve, α, β, say, is such that

$$\Sigma \frac{1}{\alpha - x_1} = 0, \quad \Sigma \frac{1}{\beta - y_1} = 0 \quad \ldots\ldots\ldots\ldots(16),$$

which give nine positions of the triple point, three of which are real. Hence, since there are six curves, namely, the inverses of the six cubics, corresponding to each triple point, there are 54 circular quartics with a triple point corresponding to a given self-apolar tetrad of foci.

In this case, it may be observed, the circular points are those points on the curve where it is intersected again by the line joining the two Hessian points of the three parameters of the triple point.

I now consider the confocal systems of circular unicursal quartics, in the case in which the four foci form a self-apolar tetrad. Using circular coordinates, if

$$\left.\begin{aligned} x &= \frac{a\lambda^3 + b\lambda^2 + c\lambda + d}{\lambda(\lambda - \alpha)} \\ y &= \frac{a'\mu^3 + b'\mu^2 + c'\mu + d'}{\mu(\mu - \alpha)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(17),$$

the general confocal and orthogonal systems are given by

$$\lambda\mu = k, \quad \lambda = k'\mu \quad \ldots\ldots\ldots\ldots\ldots(18).$$

Now the tangents at the curve parallel to x will be self-apolar, if $a\lambda^3 + b\lambda^2 + c\lambda + d$ and $\lambda(\lambda - \alpha)$ are apolar, that is, if $b\alpha + c = 0$, and similiarly for y, if $b'\alpha' + c' = 0$.

Hence, altering the origin suitably, the curve becomes

$$x = \frac{a\lambda^3 + d}{\lambda(\lambda - \alpha)}, \quad y = \frac{a'\mu^3 + d'}{\mu(\mu - \alpha')} \quad \ldots\ldots\ldots\ldots(19),$$

where $\lambda\mu = k$, $\lambda = k'\mu$ give the confocal and orthogonal

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systems of curves. Since the equations giving the foci are in this case

$$x^{-\frac{1}{2}} + \left(x - \frac{3d}{\alpha^2}\right)^{-\frac{1}{2}} + (x + 3a\alpha)^{-\frac{1}{2}} = 0,$$

$$y^{-\frac{1}{2}} + \left(y - \frac{3d'}{\alpha'^2}\right)^{-\frac{1}{2}} + (y + 3a'\alpha')^{-\frac{1}{2}} = 0 \quad \ldots\ldots(20),$$

which are of the same form as (12), we can find nodal cubics having all their foci in common with this system. Again, if $k'^3ad' = a'd$ or $k^3aa' = dd'$, we get curves of the system with a triple point at the origin.

In this case the circular points are a particular pair of points on the curve, namely, those points from which the four tangents form self-apolar tetrads. To find these points when the curve is written in the form

$$x : y : z = f_1 : f_2 : f_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots(21),$$

where $f_1 = a_1\lambda^4 + b_1\lambda^3\mu + c_1\lambda^2\mu^2 + d_1\lambda\mu^3 + c_1\mu^4,$

$f_2 = a_2\lambda^4 + \&c.,\ f_3 = a_3\lambda^4 + \&c.,$

I equate

$$lf_1 + mf_2 + nf_3 \text{ to } (\alpha\lambda + \beta\mu)\, A\lambda^3 + B\lambda^2\mu + C\lambda\mu^2 + D\mu^3) \quad \ldots\ldots\ldots(22),$$

which gives

$$A\alpha = la_1 + ma_2 + na_3,\ \alpha B + BA = lb_1 + mb_2 + nb_3,\ \&c.,$$

$$\beta D = le_1 + me_2 + ne_3,$$

from which

$$B = \frac{\alpha\Sigma lb - \beta\Sigma la}{\alpha},\quad C = \frac{\beta\Sigma ld - \alpha\Sigma le}{\beta^2} \quad \ldots\ldots(23).$$

Again, for another line passing through the point $\alpha\lambda + \beta\mu = 0$, we put

$$l'f_1 + m'f_2 + n'f_3 = (\alpha\lambda + \beta\mu)\,(\Delta'\lambda^3 + B'\lambda^2\mu + C'\lambda\mu^2 + D'\mu^3),$$

whence $A'\alpha = \Sigma l'a,\quad B' = \dfrac{\alpha\Sigma l'b - \beta\Sigma l'a}{\alpha},$ &c. $\ldots\ldots(24).$

Now, expressing that

$$A\lambda^3 + B\lambda^2\mu + C\lambda\mu^2 + D\mu^3$$

and

$$A'\lambda^3 + B'\lambda^2\mu + C'\lambda\mu^2 + D'\mu^3$$

are apolar, we have

$$3\,(AD' - A'D) = BC' - B'C,$$

which from (23) and (24) becomes

$$(\Sigma' lb\Sigma l'e - \Sigma l'b\Sigma le)\alpha^2 + (\Sigma la\Sigma l'd - \Sigma l'a\Sigma ld)\beta^2$$
$$- \{\Sigma lb\Sigma l'd - \Sigma l'b\Sigma ld - 2\,(\Sigma la\Sigma l'e - \Sigma l'a\Sigma le)\}\,\alpha\beta = 0 \ldots(25).$$

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Now, observing that the point $\alpha\lambda + \beta\mu$ is the intersection of the lines

$$lx + my + nz = 0, \quad l'x + m'y + n'z = 0,$$

we have

$$mn' - m'n \propto a_1\beta^4 - b_1\beta^3\alpha + c_1\beta^2\alpha^2 - d_1\beta\alpha^3 + l_1\alpha^4,$$

$$nl' - n'l \propto a_2\beta^4 - \&c., \quad lm' - l'm \propto a_3\beta^4 - \&c.,$$

Thus (25) becomes

$$\begin{aligned}\alpha^2 \{(aeb))\ \beta^4 + (ceb)\ \alpha^2\beta^2 - (deb)\ \alpha^3\beta\} \\ + \beta^2 \{- (bad)\ \beta^3\alpha + (cad)\ \beta^2\alpha^2 + (ead)\ \alpha^4\} \\ - \alpha\beta \{(abd)\ \beta^4 + (cbd)\ \alpha^2\beta^2 + (ebd)\ \alpha^4\} \\ + 2\alpha\beta \{- (bae)\ \beta^3\alpha + (cae)\ \beta^2\alpha^2 - (dae)\ \beta\alpha^3\} = 0,\end{aligned}$$

which, as the coefficients of $\alpha^5\beta$ and $\alpha\beta^5$ vanish identically, is divisible by $\alpha^2\beta^2$ giving

$$\begin{aligned}\{(ceb) - 3\ (dae)\}\ \alpha^2 + \{(cad) - 3\ (bae)\}\ \beta^2 \\ + \{2\ (cae) - (cbd)\}\ \alpha\beta = 0 \ldots\ldots (26).\end{aligned}$$

This is the quadratic giving the parameters of the two points on the curve from which the tangents form self-apolar tetrads. From the theory of two binary cubics we can show that we have, identically,

$$R = P^3 - 27\,Q \ldots\ldots\ldots\ldots\ldots\ldots (27),$$

where R is the sextic giving the six parameters of the three nodes, P is the quadratic (26), and Q is the sextic giving the parameters of the six points where the inflexional tangents meet the curve again. If the general trinodal quartic, referred to the triangle formed by its nodes be written

$$ay^2z^2 + bz^2x^2 + cx^2y^2 + 2xyz\ (fx + gy + hz) = 0 \ldots (28),$$

the conic

$$(gh - af)\ yz + (hf - bg)\ zx + (fg - ch)\ xy = 0 \ldots . (29)$$

passes through the two points on the curve given by (26), and the anharmonic ratio of the tangents from any point x, y, z of the curve is a function of

$$\frac{\{(gh - af)\ yz + (hf - bg)\ zx + (fg - ch)\ xy\}^2}{(xyz)^2} \ldots (30).$$

In the case of the curve (19) that we have been considering, if we use trilinear coordinates, in which case the circular points are such that $\alpha\alpha' = \beta\beta' = \gamma\gamma'$, the conic

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0 \ldots . (31)$$

must be a circle, as must also be the conic (29).

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From (30) we can easily show that if the trinodal quartic (28) is circular, namely, such that (31) is a circle, and if the line

$$(gh - af)\,\alpha + (hf - bg)\,\beta + (fg - ch)\,\gamma = 0 \ldots\ldots (32)$$

meets α, β, γ at angles whose sum is nothing, then the foci lie by fours on four circles.

If the three coefficients in the quadratic (26) all vanish, it is evident that the curve is such that the tangents from every point thereof to itself form self-apolar tetrads. From the consideration of the forms (28) and (29) we see that in this case $f = g = h = 0$, so that the curve is

$$ay^2z^2 + bz^2x^2 + cx^2y^2 = 0 \ldots\ldots\ldots\ldots (33),$$

that is, the projection of a lemniscate of Bernoulli.

This result was originally given by Casey, who showed also that the points of contact of the four tangents lie on a line. Hence, using trilinear coordinates, it appears that the foci of the curve

$$\beta^2\gamma^2 \sin 2A + \gamma^2\alpha^2 \sin 2B + \alpha^2\beta^2 \sin 2C = 0 \ldots\ldots (34)$$

form a self-apolar tetrad. This curve passes through the circular points and is the inverse with regard to the triangle of the polar circle. It may be observed that the curve determined by the equations

$$x = \frac{m + n\lambda^2}{\lambda(\lambda - \alpha)},\quad y = \frac{m' + n'\lambda^2}{\lambda(\lambda - \beta)} \ldots\ldots\ldots\ldots (35),$$

in circular coordinates is a lemniscate of the form (34) if

$$\alpha(mn' + 2m'n) = \beta(2mn' + m'n) \ldots\ldots\ldots (36),$$

and this condition will be found to give two sets of four lemniscates such as (34) belonging to the confocal and orthogonal systems (19).

I now consider the confocal and orthogonal systems of unicursal quartics with the line at infinity as a bitangent and a self-apolar tetrad of foci. Using circular coordinates the equations

$$\left.\begin{aligned} 6\lambda^2 x &= a\lambda^4 + 4b\lambda^3 + 6c\lambda^2 + 4d\lambda + e \\ 6\mu^2 y &= a'\mu^4 + 4b'\mu^3 + 6c'\mu^2 + 4d'\mu + e \end{aligned}\right\} \ldots\ldots (37),$$

where $\lambda\mu = k$, $\lambda = k'\mu$, give two confocal and orthogonal systems of unicursal quartics with the line at infinity as a bitangent. Now to determine the foci, putting $c - x = z$, we have

$$(z^2 - pz + q)^2 = (z^2 + m)^2,$$

where

$$ae + 2bd = p,\ ad^2 + eb^2 = q,\ ae - 4bd = 3m;$$

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that is,

$$(3m+2p)z^4-2qz^3+(3m^2-p^2)z+2pqz+m^2-q^2=0 \quad \ldots(38).$$

Hence, expressing that the roots of this equation are self-apolar, we obtain

$$45m^4+24pm^3-6p^2m^2-36q^2m+p^4-12pq^2=0,$$

which, being of the form

$$(3m+p)(15m^3+3pm^2-3p^2m+p^3-12q^2)=0 \quad \ldots(39),$$

is satisfied by

$$3m+p=0, \text{ or } ae-bd=0 \quad \ldots\ldots\ldots(40).$$

Similarly from the value of y in (37) we have $a'e'-b'd'=0$.

We have thus, again, corresponding to simple relations orthogonal systems of unicursal quartics with a self-apolar tetrad of foci in common. We may write them thus:

$$\left.\begin{aligned} x&=\frac{a(\lambda^4+4\lambda^3)+e(4\lambda+1)}{6\lambda^2}\\ y&=\frac{a'(\mu^4+4\mu^3)+e'(4\mu+1)}{6\mu^2}\end{aligned}\right\} \quad \ldots\ldots(41),$$

where $\lambda\mu=k$, $\lambda=k'\mu$, and k, k' are the parameters of the orthogonal system. Or, again, we can put in place of (41)

$$\left.\begin{aligned} x&=\frac{a(\lambda^4+4\lambda)+e(4\lambda^3+1)}{6\lambda^2}\\ y&=\frac{a'(\mu^4+4\mu)+e'(4\mu^3+1)}{6\mu^2}\end{aligned}\right\} \quad \ldots\ldots\ldots(42).$$

As a particular case of quartics of this kind we may notice the curve with three points of undulation. Taking area coordinates, such a curve can be written

$$x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}=0,$$

or $$(x^2+y^2+z^2-2yz-2zx-2xy)^2-128xyz(x+y+z)=0,$$

where x, y, z are the tangents at the points of undulation, and the line at infinity $x+y+z=0$ is a bitangent. The tangential equation of this curve is

$$\lambda^{-\frac{1}{2}}+\mu^{-\frac{1}{2}}+\nu^{-\frac{1}{2}}=0,$$

or $$(\lambda\mu+\mu\nu+\nu\lambda)^2-27\lambda^2\mu^2\nu^2=0.$$

Hence, if x, y are circular coordinates and a_1, b_1; a_2, b_2; a_3, b_3 are the coordinates of the vertices of the triangle formed by

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he tangents at the undulations, the tangents drawn to the curve from a circular point are

$$(x-a_1)^{-\frac{1}{3}}+(x-a_2)^{-\frac{1}{3}}+(x-a_3)^{-\frac{1}{3}}=0,$$

or $$(3x^2-2p_1x+p_2)^3=27\,(x^3-p_1x^2+p_2x-p_3)^2=0,$$

so that taking the origin at the centre of gravity, which gives $p_1=0$, we get, for the foci, if $p_2=3a$, $p_3=b$,

$$3ax^4-2bx^3+6a^2x^2-6abx+b^2-a^3=0;$$

and since the invariant S vanishes for this equation, it follows that the foci form a self-apolar tetrad.

With regard to the unicursal curve of the sixth order, the reciprocal of a quartic with a triple point, I observe that it involves ten constants, so that if the four foci are given, there are still two indeterminate constants.

Taking circular coordinates, if the equation of a line is $\alpha x+\beta y+\gamma=0$, the equations

$$\rho\alpha=l\lambda f_1,\ \rho\beta=m\mu f_2,\ \rho\gamma=\lambda\mu\,\frac{(f_1-f_2)}{\lambda-\mu}\quad\ldots\ldots(43),$$

where

$$f_1=Am^3(\lambda-\mu)^3+Bm^2\lambda\,(\lambda-\mu)^2+Cm\gamma^2(\lambda-\mu)+\lambda^3,$$
$$f_2=A'l^3(\lambda-\mu)^3+B'l^2\mu\,(\lambda-\mu)^2+c'l\mu^2(\lambda-\mu)+\mu^3,$$

represent tangentially the reciprocal of a quartic with a triple point, having a focus at the origin and three other foci given by

$$\left.\begin{aligned}x^3+Cx^2+Bx+A=0\\ y^3+Cy^2+By+A'=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(44).$$

The constants l, m are thus the two constants which remain indeterminate when the four foci are given. Hence, from (44), the four foci form a self-apolar tetrad, if $B^2=3AC$, $B'^2=3A'C'$. If the three tangents at the triple point coincide, the reciprocal curve is of the fourth degree with two cusps and a double point and a point of undulation. Such a curve can be written

$$(xy+yz+zx)^3-16z^3xy=0\quad\ldots\ldots\ldots\ldots(45),$$

where xz, yz are the cusps, xy is the node, and $x=y=2z$ is the point of undulation. If S, T are the invariants of the four tangents drawn from x, y, z to this curve, it can be shown that

$$\frac{S^3}{27T^2}=\frac{16z^3\,(4z-x-y)}{(xy+yz+zx-8z)^2}\quad\ldots\ldots\ldots(46).$$

Hence we see that the tangents drawn to the curve from any point on z, the line joining the cusps, are such that their

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invariant S vanishes, so that if the cusps are at infinity the four foci of the curve form a self-apolar tetrad. This curve may be expressed in circular coordinates thus

$$x = \frac{(a\theta^2 + e)(\theta^2 + 4\theta + 1)}{\theta^2}, \quad y = \frac{(a'\theta^2 + e')(\theta^2 + 4\theta + 1)}{\theta^2} \quad \ldots\ldots\ldots(47),$$

where the origin is the node. It is thus shown to be a particular curve of the system (41).

I may observe that, the locus of points whence tangents to the reciprocal of a quartic with a triple point (43) are self-apolar being a conic, this conic must be a circle when the foci form a self-apolar tetrad.

Further I notice here, as a particular case of (43), that the tricuspidal quartic defined tangentially by

$$\rho\alpha = l\lambda f_1, \ \rho\beta = m\mu f_2, \ \rho\gamma = \lambda\mu \frac{(f_1 - f_2)}{\lambda - \mu} \ldots\ldots (48),$$

where
$$f_1 = Am^2(\lambda - \mu)^2 + Bm\lambda(\lambda - \mu) + \lambda^2,$$
$$f_2 = A'l^2(\lambda - \mu)^2 + B'l\mu(\lambda - \mu) + \mu^2,$$

has its foci given by

$$\left.\begin{aligned} x(x^2 + Bx + A) &= 0 \\ y(y^2 + B'y + A') &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots(49),$$

Of course the expression of this curve directly can be obtained from (48) by taking the Jacobians of the three binary cubics to which α, β, γ are proportional, when we have $x : y : 1 = J_{23} : J_{31} : J_{12}$ in circular coordinates.

I observe that if $B^2 = 3A$, $B'^2 = 3A'$ in (49) the three foci form an equilateral triangle.

From (48) we can obtain the equation of the confocal system of cuspidal cubics. We express that (48) is a cuspidal cubic by making the parameters of the points of contact of the bitangent coincide. For the latter points $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\gamma}$ have the same values, respectively. Hence, if

$$\lambda = \theta, \ \mu = 1, \ Am^2 - A'l^2 + Bm + 1 = p,$$
$$A'l^2 - Am^2 - B'l + 1 = q,$$

we must have, for the point of inflexion,

$$\frac{d}{d\theta}\left\{\frac{Am^2(\theta - 1)^2 + Bm\theta(\theta - 1) + \theta^2}{p\theta + q}\right\} = 0,$$
$$\frac{d}{d\theta}\left\{\frac{A'l^2(\theta - 1)^2 + B'l(\theta - 1) + 1}{\theta(p\theta + q)}\right\} = 0,$$

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and these give two quadratics in θ, from which, by the elimination of θ, we obtain a relation connecting l, m. The equations (48), then, subject to this relation give the singly infinite system of cuspidal cubics with foci determined by (49).

The simple form assumed by the system (41) or (42) with a self-apolar tetrad of foci depends upon the algebraic theorem that, if we take the discriminant of $U+\lambda P^2$, where U is a binary quartic and P is a quadratic, the condition that the resulting biquadratic in λ should be self-apolar has for a factor $3\Delta'-\Delta S$, where Δ' is the discriminant of the quadratic obtained by substituting differential symbols in P and operating on U, Δ is the discriminant of P, and S is the invariant of U of the second degree in the coefficients. We may mention another case in which we can have a confocal system of unicursal quartics with a self-apolar tetrad of foci, but in which there is no such algebraical simplicity. Consider a unicursal quartic of which the line at infinity is an inflexional tangent, then the equations in circular coordinates:

$$\left.\begin{aligned} 4\lambda x &= a\lambda^4 + 4b\lambda^3 + 6c\lambda^2 + 4d\lambda + e \\ 4\mu y &= a'\mu^4 + 4b'\mu^3 + 6c'\mu^2 + 4d'\mu + e' \end{aligned}\right\} \quad \ldots\ldots(50),$$

where $\lambda = k\mu$, represent a system of unicursal quartics with their four foci in common, k being the parameter of the confocal system. In this case the conditions that the foci should form a self-apolar tetrad are

$$36\,(b^2c^2 + a^2ce - ac^3 - aeb^2)^2$$
$$= (3ac^2 - 4b^2c + a^2e)\,\{a\,(ae + 3c^2)^2 - 8b^2\,(c^3 + 3aec - 2eb^2)\} \ldots(51),$$

and a similar relation between a', b', c', e'.

It may be observed that for no other relation connecting four foci is there the same algebraic simplicity. We may notice, however, a peculiarity for bicircular quartics, in the case in which the pencils joining the four foci to the circular points are harmonic. Let these pencils in circular coordinates be

$$\left.\begin{aligned} X &= m^4\,(x-a)^4 - n^4\,(x-a')^4 = 0 \\ Y &= m'^4\,(y-b)^4 - n'^4\,(y-b')^4 = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots(52),$$

then it is evident that the intersections of these pencils lie on the eight circles

$$\left.\begin{aligned} mm'\,(x-a)\,(y-b) - enn'\,(x-a')\,(y-b') &= 0 \\ mn'\,(x-a)\,(y-b') - e'm'n\,(x-a')\,(y-b) &= 0 \end{aligned}\right\} \quad \ldots(53),$$

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where $\epsilon^4 = \epsilon'^4 = 1$. Four of these circles selected so as to be mutually orthogonal are the Jacobian circles of bicircular quartics with $X=0$, $Y=0$ as foci, namely, the fixed circles cutting orthogonally the four systems of generating circles of the quartics. Then the four other circles are Jacobian circles of another system of bicircular quartics, cutting the first system at an angle of 45°, the differential equations being

$$\left.\begin{aligned} mn\,(a-a')\,\frac{dx}{\sqrt{X}} \pm m'n'\,(b-b')\,\frac{dy}{\sqrt{Y}} &= 0 \\ mn\,(a-a')\,\frac{dx}{\sqrt{X}} \pm m'n'\,(b-b')\,\frac{\sqrt{(-1)}\,dy}{\sqrt{Y}} &= 0 \end{aligned}\right\} \ldots(54).$$

I now proceed to mention a case in which the foci of a cubic U form two triangles and the foci of the Hessian H, which is circular, are the points, which with the triangles, respectively, form self-apolar tetrads. A cubic U being written in the form

$$U = ax^3 + by^3 + cz^3 + du^3 = 0 \ldots\ldots\ldots\ldots(55),$$

where $x+y+z+u=0$, identically, the point xy is a point on the Hessian whose equation is

$$\frac{1}{ax} + \frac{1}{by} + \frac{1}{cz} + \frac{1}{du} = 0 \ldots\ldots\ldots\ldots(56).$$

We can then easily show that the tangents from xy to U are

$$\left.\begin{aligned} (\sqrt{c}+\sqrt{d})^2\,(ax^3+by^3) - cd\,(x+y)^3 &= 0 \\ (\sqrt{c}-\sqrt{d})^2\,(ax^3+by^3) - cd\,(x+y)^3 &= 0 \end{aligned}\right\} \ldots\ldots(57),$$

and the tangents from xy to H are

$$\frac{1}{ax} + \frac{1}{by} - \frac{(\sqrt{c}+\sqrt{d})}{cd\,(x+y)} = 0, \quad \frac{1}{ax} + \frac{1}{by} - \frac{(\sqrt{c}-\sqrt{d})^2}{cd\,(x+y)} = 0 \ldots(58).$$

Now the latter quadratics are the Hessian covariants of the binary cubics (57), respectively. Hence taking circular coordinates and making the Hessian a circular cubic, if a triad of tangents to U from a circular point I be written

$$m\,(x-a)^3 + n\,(x-a')^3 = 0 \ldots\ldots\ldots\ldots(59),$$

the corresponding tangents to H are

$$(x-a)\,(x-a') = 0 \ldots\ldots\ldots\ldots(60),$$

and the same is the case for the other triad of tangents from I, and this proves what we have stated above.

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It may be observed that the two triads satisfy the combinantive invariant relation $Q=0$ (Salmon's *Higher Algebra*, Art. 199) with each other.

I may conclude by noticing a case in which the three foci of a tricuspidal quartic form a self-apolar tetrad with a point on the double tangent of the curve. It can easily be shown that the tangents from a point P to the tricuspidal quartic

$$\sqrt{(yz)}+\sqrt{(zx)}+\sqrt{(xy)}=0 \quad \ldots\ldots\ldots\ldots (61),$$

together with the line from P to the point l, m, n on the double tangent $x+y+z=0$, where $l+m+n=0$, form a self-apolar tetrad, if P lies on the conic

$$mnyz+nlzx+lmxy=0 \quad \ldots\ldots\ldots\ldots (62),$$

which passes through the cusps and touches the quartic.

Hence putting $l\alpha$, $m'\beta$, $n'\gamma$, respectively, for x, y, z, where α, β, γ are trilinear coordinates, and making the conic (62) coincide with the circle passing through the cusps, we see that the foci of the tricuspidal quartic

$$\sqrt{\left(\frac{l\sin A}{\alpha}\right)}+\sqrt{\left(\frac{m\sin B}{\beta}\right)}+\sqrt{\left(\frac{n\sin C}{\gamma}\right)}=0 \ldots (63),$$

where $l+m+n=0$ form with the point

$$l^2\sin A,\ m^2\sin B,\ n^2\sin C \quad \ldots\ldots\ldots\ldots (64),$$

a self-apolar tetrad of points with regard to I, J. The quartic (63) touches the circle through the cusps, and the point (64) is the point of contact of the double tangent

$$\frac{\alpha}{l\sin A}+\frac{\beta}{m\sin B}+\frac{\gamma}{n\sin C}=0$$

with the conic

$$\sqrt{\left(\frac{\alpha}{\sin A}\right)}+\sqrt{\left(\frac{\beta}{\sin B}\right)}+\sqrt{\left(\frac{\gamma}{\sin C}\right)}=0.$$

II. *On the foci of certain curves of the third and fourth order.*

I propose to determine here the foci of the cubic with the line at infinity as an inflexional tangent, and of the bicuspidal quartic subject to certain conditions. In all the cases considered the curve has four foci. Now, to find the foci of a curve, the obvious plan suggests itself to investigate the equation of a confocal curve. In each case, then, we should seek the equation of the simplest curve having the same number of foci as the given curve, and this being obtained,

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we may consider the problem as solved. Of course, this does not refer to special cases in which there may be simpler solutions owing to the foci being connected by certain relations. Now the nodal cubic, involving eight constants, is the simplest curve with four foci, and if the four foci lie on a circle the simplest curve is a circular cubic or a bicircular quartic. Now I observe that the cubic with the line at infinity as an inflexional tangent involves seven constants, so that the four foci are connected by a single relation, which, however, does not seem to be capable of expression in a simple form.

In order to obtain the equation of a confocal curve I make use of a theorem that I have stated elsewhere, namely, that in general the locus of the foci of a system of curves involving a single parameter is confocal with the envelope of the system. I thus consider the system of parabolæ

$$t\{t(ax+by)+cx+dy\}^2 = lx+my+n \ldots\ldots\ldots(1)$$

expressed in circular coordinates. The envelope of this system is

$$27(ax+by)(lx+my+n)+4(cx+dy)^3=0 \ldots(2),$$

which represents a cubic with the line at infinity as an inflexional tangent, $ax+by$ and $lx+my+n$ as inflexional tangents, and $cx+dy$ the line passing through the three corresponding points of inflexion. Now, for the focus of (1), we have

$$\left.\begin{aligned} x &= \frac{m^2+4nt(d+bt)^2}{4t(d+bt)(p+qt)} \\ y &= -\frac{n^2+4mt(c+at)^2}{4t(c+at)(p+qt)} \end{aligned}\right\} \ldots\ldots\ldots\ldots(3),$$

where $mc-ld=p$, $ma-lb=q$. This represents a unicursal quartic passing through the circular points at infinity, so that we have thus found a circular unicursal quartic confocal with the given curve. Now let us make the substitution $t=\frac{d}{k\theta-b}$ in the value of x in (3), and we get

$$x=\frac{m^2(k\theta-b)^3+4nk^3d^3\theta^2}{4kd^3\theta(f+g\theta)} \ldots\ldots\ldots\ldots(4),$$

where $g=kp$, $f=dq-bp=m(ad-bc)$;

also, making the substitution $t=\frac{c}{k'\phi-a}$ in the value of y, we get

$$y=-\frac{\{l^2(k'\phi-a)^3+4nk'^3c^3\phi^2\}}{4k'c^3\phi(f'+g'\phi)} \ldots\ldots\ldots\ldots(5),$$

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where $g'=k'p$, $f'=l(ad-bc)$. Now these transformations do not affect the position of the foci, as they leave the discriminants unaltered. Hence, considering the confocal curve given by $\phi=\theta$, and determining k, k', so that $fg'-f'g=0$, we obtain $lk-mk'=0$. Thus, taking $k=m$, $k'=l$, we have the confocal curve

$$\left.\begin{aligned} x &= \frac{(m\theta-b)^3+4nd^3\theta^2}{4d^3\theta\{ad-bc+(mc-ld)\theta\}} \\ y &= -\frac{(l\theta-a)^3+4nc^3\theta^2}{4c^3\theta\{ad-bc+(mc-ld)\theta\}} \end{aligned}\right\} \ldots\ldots\ldots(6),$$

which represents a nodal cubic. Now I observe that from (2) there is no loss of generality in the expression of the given cubic if we take $c=d=1$. The three points at infinity on the cubic (6) are then given by $\theta=0$, ∞, and $\frac{a-b}{l-m}$ respectively. We thus easily find that the asymptotes of (6) are

$$\left.\begin{aligned} x+y &= \frac{3(ma-lb)^2}{4(l-m)(a-b)} \\ a^3x+b^3y &= \frac{3a^2b^2(ma-lb)}{4(a-b)} \\ l^3x+m^3y &= \frac{3l^2m^2(ma-lb)+4n(l^3-m^3)}{4(m-l)} \end{aligned}\right\} \ldots(7),$$

also for the node we have

$$\left.\begin{aligned} x &= \frac{(b^3-km^3)^3+4nkb^3}{4k\{km^3(a-b)+b^3(m-l)\}} \\ y &= \frac{(kl^3-a^3)^3-4nka^3}{4k\{kl^3(a-b)+a^3(m-l)\}} \end{aligned}\right\} \ldots\ldots\ldots(8),$$

where

$$k=\frac{a^2+b^2+ab}{l^2+m^2+lm},$$

and the line of inflexions δ is parallel to

$$\{4n(a-b)+3la(lb-ma)\}x + \{4n(a-b)+3mb(lb-ma)\}y=0 \ldots\ldots(9),$$

and there is no difficulty in finding any lines or points fundamentally connected with the cubic (6).

Now suppose we want to find the condition that the foci of (2) should lie on a circle, we have to express that the foci of (6) are concyclic. This condition I have shown elsewhere

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is satisfied in the case of a nodal cubic U, if $U+\lambda\delta^2=0$ is capable of being made a circular cubic, where δ is the line of inflexions and λ is a parameter. But we have the asymptotes of U from (7) and the direction of δ from (9). Thus, taking only the highest terms,

$$(x+y)(a^2x+b^2y)(l^2x+m^2y)+\lambda\{(k+la)x+(k+mb)y\}^2,$$

where $$k=\frac{4n(a-b)}{3(lb-ma)},$$

must be divisible by xy.

This gives

$$\left(\frac{k+la}{k+mb}\right)^3=\frac{l^2a^2}{m^2b^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

As a particular case $la=mb$, and then $x-y=0$ is an axis of symmetry on which four foci are situated. Omitting, then, this factor we get

$$k=\frac{(\theta^2-\theta)\,lmab}{\theta la-\theta^2 mb},$$

where θ is a cube root of unity. Now making

$$a=ce^{i\alpha},\quad b=ce^{-i\alpha},\quad l=e^{i\beta},\quad m=e^{-i\beta},\quad n=-2p,$$

we get $$k=\frac{8p}{3}\,\frac{\sin\alpha}{\sin(\alpha-\beta)}=\frac{c\sqrt{3}}{2\sin\left(\alpha+\beta+\frac{2\pi}{3}\right)},$$

whence $16p\sin\alpha\sin(\alpha+\beta+\frac{2}{3}\pi)=3c\sqrt{3}\sin(\alpha-\beta)$,

or $$16\delta\sin\alpha\sin\beta\sin(\alpha+\beta+\tfrac{2}{3}\pi)-3c\sqrt{3}\sin(\alpha-\beta)=0 \quad \ldots\ldots\ldots (11),$$

where δ is the distance between the inflexions

$$ax+by=x+y=0,\quad lx+my+n=x+y=0,$$

and α, β are the angles $x+y=0$ makes with the curve at these inflexions.

From the equation (1) we see that the cubic (2) is generated as the envelope of a system of parabolæ touching a fixed line $L=lx+my+n=0$, the two points of contact with the envelope being collinear with the fixed point

$$ax+by=x+y=0.$$

Now we could evidently generate the cubic (2) as the envelope of a parabola obtained by interchanging

$$ax+by \quad \text{and} \quad lx+ny+n.$$

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In this case, in order to preserve the same form, we should have to transfer the origin to the point given by

$$x' + y' = 0, \quad lx' + my' + n = 0, \quad n' = ax' + by'.$$

Now, transforming the nodal cubic (6) to this origin, we have for x

$$x = \frac{(m\theta - b)^2 + 4n\theta^2}{4\theta\{a - b + (m - l)\theta\}} + \frac{n}{l - m}$$
$$= \frac{(m\theta - b)^2 + 4n'\theta}{4\theta\{a - b + (m - l)\theta\}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(12),$$

which, by the substitution of $\frac{1}{\theta}$ for θ, and l, m, a, b for a, b, l, m respectively, simply reproduces the same confocal nodal cubic (6).

Now it is to be observed that a cubic with the line at infinity as an inflexional tangent can be written in the form (2) in four ways. The cubic can be written in the unique canonical form

$$(\lambda x + \mu y)^2 = x^3 + fx + g \quad \ldots\ldots\ldots\ldots\ldots(13),$$

where, the coordinates being rectangular, $\lambda x + \mu y = 0$ is the harmonic polar of the point of inflexion at infinity, and the origin is at the centroid of the three points in which the harmonic polar meets the curve. But if we put

$$x^3 + fx + g = (x - h)^3 + (\alpha + \beta x)^2 \quad \ldots\ldots\ldots(14),$$

we have

$$3h = \beta^2, \quad 3h^2 + 2\alpha\beta = f, \quad \alpha^2 - h^3 = g \quad \ldots\ldots(15),$$

whence h is determined by the biquadratic

$$3h^4 + 6fh^2 + 12gh - f^2 = 0 \quad \ldots\ldots\ldots\ldots(16).$$

Now, using circular coordinates, we can write the cubic (13) in one of the four forms (14) determined by (16), namely,

$$(px + qy)^2 = (x + y - h)^3 + \{\alpha + \beta(x + y)\}^2,$$

that is

$$\{px + qy + \alpha + \beta(x + y)\}\{px + qy - \alpha - \beta(x + y) = (x + y + h)^3,$$

which, by transference to the origin determined by

$$x' + y' = h, \quad px' + qy' = \alpha + \beta h,$$

becomes

$$\{px + qy - \beta(x + y)\}\{px + qy + \beta(x + y) + 2(\alpha + \beta h) = (x + y)^3$$
$$\ldots\ldots\ldots(17),$$

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which is of the form (2). For this cubic, then, if we take the confocal nodal cubic (6), it is easy to see that we obtain a different curve for each value of h and form (17). We have thus found four nodal cubics confocal with a given cubic of which the line at infinity is an inflexional tangent.

It may be observed that from (6) we deduce that, if we write

$$\left.\begin{aligned} x &= \frac{(m\theta - b)^3 + 4nd^3\theta^2}{4d^3\theta\{ad - bc + (me - ld)\theta\}} \\ y &= \frac{(a - l\phi)^3 - 4nc^3\phi^2}{4c^3\phi\{ad - bc + (mc - ld)\phi\}} \end{aligned}\right\} \quad \ldots\ldots(18),$$

then the equations $\phi = \lambda\theta$, $\phi\theta = \mu$, where λ, μ are parameters, represent systems of circular unicursal quartics cutting each other orthogonally, and confocal with each other and with the cubic (2).

I now proceed to consider the bicuspidal quartic of which the bitangent coincides with the line at infinity. Such a curve is determined by eight constants and has four finite foci. It would seem, then, that there ought to be a definite number of curves corresponding to four given foci. I shall show, however, that this is not the case—that the foci are not arbitrarily placed, but are connected by a certain relation, and that there is a singly infinite system of bicuspidal quartics with given foci.

Let us consider the system of conics

$$ax^2 + by^2 + 2gx + 2fy + c + 2\theta(lx + my + n) = \theta^2 = 0 \ldots(19)$$

in rectangular coordinates, where a, b, &c., are constants and θ is a parameter. The envelope of the system is

$$27(ax^2 + by^2 + 2gx + 2fy + c)^2 + 32(lx + my + n)^3 = 0 \ldots(20),$$

which represents a quartic with the line at infinity as bitangent, and with cusps at the two points

$$ax^2 + by^2 + 2gx + 2fy + c = 0, \quad lx + my + n = 0.$$

Hence, if we take the locus of the foci of (19), we obtain a curve having the same foci as (20). Now the finding of the foci of (19) is simplified by the fact that the axes of coordinates are parallel to the axes of the curve. Writing (19) in the form

$$aX^2 + bY^2 + \Theta = 0 \ldots\ldots(21),$$

where $aX = ax + g + \theta l$, $bY = by + f + \theta m$,

$$\Theta = \theta^2 - \frac{1}{a}(g + \theta l)^2 - \frac{1}{b}(f + \theta m)^2 + 2n\theta + c \ldots(22),$$

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we have for the foci on $Y=0$

$$by+f+\theta m=0, \quad (ax+g+\theta l)^2=\frac{a}{b}(a-b)\,\Theta\ldots(23).$$

Eliminating θ from the equations (18), we obtain an equation of the form

$$\left\{ax+g-\frac{l}{m}(by+f)\right\}^2=Ay^3+By^2+Cy+D\ldots(24).$$

This represents a cubic with the line at infinity as an inflexional tangent, and the line $m(ax+g)-l(by+f)=0$ as the corresponding harmonic polar. Taking the origin on the latter line we may put $g=kl, f=km$.

Again, fixing the origin at the centroid of the three points where the harmonic polar meets the curve, we get

$$-3k=\frac{l^2}{a}+\frac{m^2}{b}\ldots\ldots\ldots\ldots\ldots\ldots(25).$$

Now, making $a=pl$, $b=qm$, where p, q are supposed to be given quantities, the cubic (24) becomes

$$\left(x-\frac{q}{p}y\right)^2=\left(\frac{pl-qm}{pqlm}\right)\{\theta^3-3k(\theta+k)^2+2n\theta+c\},$$

where $k+qy+\theta=0$; that is, the cubic is

$$\left(x-\frac{q}{p}y\right)^2=\rho^2\{q^3y^3+(3k^2+2n)qy+k^3+2nk-c\}\ldots(26),$$

where

$$\frac{qm-pl}{pqlm}=\rho^2\ldots\ldots\ldots\ldots\ldots\ldots(27).$$

Hence, if the right-hand expression in (21) is

$$q^3y^3+\alpha qy+\beta,$$

we have

$$2n=\alpha-3k^2,$$

$$c=k^3+2nk-\beta=-2k^3+\alpha k-\beta\ldots\ldots\ldots(28).$$

Thus the quartic (15) becomes

$$27\{lpx^2+mqy^2+2k(lx+my)-2k^3+\alpha k-\beta\}^2$$
$$+4(2lx+2my+\alpha-3k^2)^3=0\ldots\ldots(29),$$

where l, m are determined in terms of k by the equations

$$\frac{1}{lp}-\frac{1}{mq}=\rho^2, \quad \frac{l}{p}+\frac{m}{q}=-3k\ldots\ldots\ldots(30),$$

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and k is an arbitrary parameter. If we make $\frac{l}{m}=t$ the parameter, we have

$$l=\frac{q-pt}{\rho^2 pq},\quad m=\frac{q-pt}{\rho^2 pqt},\quad k=\frac{(pt-q)(p+qt)}{3p^2q^2\rho^2 t}\ldots(31).$$

We have thus found a singly infinite system of bicuspidal quartics (29), with the line at infinity as bitangent, having all their foci in common with a fixed cubic (26), having the line at infinity as inflexional tangent. The four foci of the quartic are thus connected by a single relation, namely, the relation connecting the foci of a cubic with the line at infinity as an inflexional tangent.

It is to be observed that if, in determining the foci of (19), instead of taking $Y=0$ in (16) and (18), we took $X=0$, we should find, as the locus of the foci, instead of (26), the cubic

$$\left(y-\frac{p}{q}x\right)^3=\rho''^2\{p^3x^3+(3k^2+2n)px+k^3+2nk-c\}\ldots(32),$$

where $$\rho''^2=\frac{pl-qm}{pqlm}=-\rho^2.$$

Thus it is easily seen that if this curve were given, we should determine from it precisely the same confocal system (24) as we have already determined from (21). Hence we see that there are two cubics, with the line at infinity as an inflexional tangent, having their four foci in common with the system of bicuspidal quartics determined by (24) and (26). Both these cubics, it may be observed, have the line $px-qy=0$ as a harmonic polar.

Since the coefficient of k^6 vanishes in (29) the highest term of the parameter is of the same degree as k^4l; that is, from (26), the degree of the highest term in t is nine.

We see thus that nine confocal curves of the system (24) can be drawn to pass through a given point.

I now propose to consider another singly infinite system of confocal bicuspidal quartics. Let the curve have an axis of symmetry and pass through the circular points, then it must have four foci on the axis of symmetry. The equation of the curve in this case may be written

$$(y^2+ax^2+2bx+c)^2-m^2(x-\alpha)^3(x-\beta)=0\ldots(32)$$

in rectangular Cartesian coordinates, where $y=0$ is the axis of symmetry. Since the curve passes through the circular points, we have $a=m+1$, so that (32) involves five independent constants.

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Hence when the four foci are given, there is still an indeterminate parameter. To find the foci then we express that

$$(X-x)^2+y^2=0$$

touches the curve; that is, we take the discriminant with regard to x of

$$\{mx^2+2x(X+b)+c-X^2\}^2-m^2(x-\alpha)^2(x-\beta)=0\ldots(33);$$

which, if we put $x=\dfrac{\beta-\alpha t^2}{1-t^2}$, and divide out by $1-t$, becomes

$$\{X^2-2\alpha X-(m\alpha^2+2b\alpha+c)\}(t^2+t^2)-\{X^2-2\beta X$$
$$+m(\alpha^2-2\alpha\beta)-2b\beta-c\}(t+1)+m(\alpha-\beta)^2=0\ldots(34),$$

and if this be written

$$p(t^2+t^2)-q(t+1)+m(\alpha-\beta)^2=0,$$

the discriminant is

$$p\{2p-18q+27m(\alpha-\beta)^2\}^2-4(p+3q)^3=0\ldots(35),$$

which being divisible by the square of infinity is of the fourth degree in x. We see thus, that if the biquadratic giving the foci is written

$$(x^2-px+q)(x^2-mx+n)^2-(x^2-\lambda x+\mu)^2\ldots(36),$$

then by identification of x^2-px+q, x^2-mx+n, $x^2-\lambda x+\mu$, with p, $\frac{1}{18}\{18q-2p-27m(\alpha-\beta)^2\}$, $\frac{1}{4}(p+3q)$, respectively, we determine the quartic (32) completely. But a given biquadratic can be written in the form (36) in a singly infinite number of ways, as there are six constants subject only to the condition $p+2m=3\lambda$. Equating (36) to $-\nu^2\phi(x)$, where $\phi(x)$ gives the four foci, if x_1, x_2 are the roots of $x^2-mx+n=0$, we have

$$x_1^2-\lambda x_1+\mu=\nu\sqrt[3]{\{\phi(x_1)\}},\quad x_2^2-\lambda x_2+\mu=\nu\sqrt[3]{\{\phi(x_2)\}},$$
$$3(2x_1-\lambda)=\frac{\nu\phi'(x_1)}{\{\phi(x_1)\}^{\frac{2}{3}}},\quad 3(2x_2-\lambda)=\frac{\nu\phi'(x_2)}{\{\phi'(x_2)\}^{\frac{2}{3}}}\ldots(37);$$

whence, eliminating λ, μ, ν, we obtain

$$(x_1-x_2)\left\{\frac{\phi'(x_1)}{\{\phi(x_1)\}^{\frac{2}{3}}}+\frac{\phi'(x_2)}{\{\phi'(x_2)\}^{\frac{2}{3}}}\right\}6\{\sqrt[3]{\phi(x_1)}-\sqrt[3]{\phi(x_2)}\},$$

$$\text{or}\quad\frac{\{(x_1-x_2)\phi'(x_1)-6\phi(x_1)\}^3}{\{\phi(x_1)\}^2}=\frac{\{(x_2-x_1)\phi'(x_2)-6\phi(x_2)\}^3}{\{\phi(x_2)\}^2}$$
$$\ldots\ldots\ldots\ldots(38),$$

namely, a relation of the twentieth degree connecting x_1, x_2.

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From (37) then we can express λ, μ, and also p, q uniquely in terms of x_1, x_2, so that the curve (32) can be expressed in terms of x_1, x_2, these parameters being connected by the relation (38). It may be observed that, if $\phi(x)$ has a square factor, the curve has an inflexional focus, and there is a singly infinite confocal system. Also, if the coefficient of x^4 in $\phi(x)$ vanishes, the curve touches the line at infinity and has three foci, and there is a singly infinite confocal system.

It may be observed that the quartic (32) can be written $S_1 S_2^2 = S_3^3$, where S_1, S_2, S_3 are circles with their centres on a line.

III. *On the determination of the foci of the plane nodal cubic in a certain case.*

I propose to show here how the foci of a nodal cubic may be determined in a certain case, namely, when the four real foci are concyclic. In a communication to the *Proceedings of the London Mathematical Society* (Vol. XII., p. 99), I showed that the anharmonic ratio of the tangents drawn from a point $P(x, y, z)$ to the nodal cubic

$$(x+y+z)^3 - 27xyz = 0 \quad \ldots\ldots\ldots\ldots\ldots(1)$$

is equal to the anharmonic function of the cubic of the system

$$(x+y+z)^3 - \lambda xyz = 0 \quad \ldots\ldots\ldots\ldots\ldots(2)$$

(λ being a parameter), which passes through P, that is, as follows from the known theory of the non-singular cubic, that the anharmonic ratio of the tangents drawn from P to the cubic (1) is equal to the anharmonic ratio of the four tangents drawn to the cubic (2) from P, including the tangent at P. I observe that the values of the invariants of either of the pencils are

$$\left.\begin{aligned} S &= 3(x+y+z)\{(x+y+z)^3 - 24xyz\} \\ T &= -\{(x+y+z)^6 - 36xyz(x+y+z)^3 + 216x^2y^2z^2\} \end{aligned}\right\} \ldots(3).$$

Hence, it follows that if a cubic of the system (2) pass through the circular points, then by the definition of a focus and the anharmonic property of a conic, the 16 foci real and imaginary of the cubic (1) lie by fours on four circles (Salmon's *Higher Plane Curves*, Art. 168). Now, putting $l\alpha$, $m\beta$, $n\gamma$ for x, y, z, respectively, where α, β, γ are trilinear coordinates, the cubic (2) may be written

$$\alpha\beta\gamma - \lambda'\delta^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots(4),$$

where δ is the perpendicular from a point on the line

$$l\alpha + m\beta + n\gamma = 0,$$

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and if this cubic pass through the circular points, namely, the points

$$\alpha : \beta : \gamma : \delta = e^{\pm i\theta_1} : e^{\pm i\theta_2} : e^{\pm i\theta_3} : e^{\pm i\theta_4},$$

where θ_1, θ_2, θ_3, θ_4 are the angles α, β, γ, δ make with a fixed line, we must have

$$\lambda' = 1, \quad \theta_1 + \theta_2 + \theta_3 = 3\theta_4 \quad \ldots\ldots\ldots\ldots\ldots(5),$$

the latter of which equations expresses that the right line passing through the three points of inflexion meets the curve at angles whose sum vanishes. Thus, when this condition is satisfied, the 16 real and imaginary foci of the curve lie by fours on four circles, but it is to be observed that this is not necessarily the case conversely, for from (3) the anharmonic ratios of the pencils of the tangents from the circular points are equal, if

$$\frac{\lambda_1 (\lambda_1 - 24)^3}{(\lambda_1^2 - 36\lambda_1 + 216)^2} = \frac{\lambda_2 (\lambda_2 - 24)^3}{(\lambda_2^2 - 36\lambda_2 + 216)^2} \quad \ldots\ldots(6),$$

in which we need not necessarily take $\lambda_1 = \lambda_2$.

I consider here, however, the simple case in which (5) is satisfied, and the equation

$$\alpha\beta\gamma\,(l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C)^{\frac{1}{2}} - (l\alpha + m\beta + n\gamma)^3 = 0 \ldots\ldots(7),$$

or $$\alpha\beta\gamma\,\{l\cos(\theta_4 - \theta_1) + m\cos(\theta_4 - \theta_2) + n\cos(\theta_4 - \theta_3)\}^3 - (l\alpha + m\beta + n\gamma)^3 = 0,$$

represents a circular cubic, the curve itself being

$$(l\alpha)^{\frac{1}{3}} + (m\beta)^{\frac{1}{3}} + (n\gamma)^{\frac{1}{3}} = 0 \ldots\ldots\ldots\ldots\ldots(8).$$

Now the fact that the tangents from a point P to the cubic (1) have the same anharmonic ratio as the tangents from P to the cubic of the system (2) passing through P, shows that the two sets of tangents are homographically connected, and I proceed to find the precise nature of this homographic relation.

Let x, y be two lines passing through P, and let z be the line passing through the inflexions of the cubics (1) and (2), and let

$$z + u_1, \quad z + u_2, \quad z + u_3,$$

where $\quad u_1 = a_1 x + b_1 y, \quad u_2 = a_2 x + b_2 y, \quad u_3 = a_3 x + b_3 y,$

be the three inflexional tangents, then if

$$pu_1 + qu_2 + ru_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots(9),$$

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identically, the cubics (1), (2) are, respectively,

$$\{p(z+u_1)\}^{\frac{1}{3}} + \{q(z+u_2)\}^{\frac{1}{3}} + \{r(z+u_3)\}^{\frac{1}{3}} = 0 \quad \ldots(10),$$

$$(z+u_1)(z+u_2)(z+u_3) = z^3 \quad \ldots\ldots\ldots(11).$$

Now to find the tangents drawn from xy to these cubics, we take the discriminants with regard to z. We thus obtain

$$\left(\frac{u_2-u_3}{p}\right)^{-\frac{1}{3}} + \left(\frac{u_3-u_1}{q}\right)^{-\frac{1}{3}} + \left(\frac{u_1-u_2}{r}\right)^{-\frac{1}{3}} = 0 \ldots\ldots(12),$$

$$(u_1)^{-\frac{1}{3}} + (u_2)^{-\frac{1}{3}} + (u_3)^{-\frac{1}{3}} = 0 \quad \ldots\ldots\ldots(13),$$

respectively. But these two equations for different values of the variable $x:y$ will evidently coincide, if we have

$$u_2-u_3 : u_3-u_1 : u_1-u_2 = pu_1' : qu_2' : ru_3',$$

which is equivalent to the single relation

$$pu_1u_1' + qu_2u_2' + ru_3u_3' = 0 \quad \ldots\ldots\ldots(14),$$

and this shows that the roots of the binary quartic (12) are, in a certain order, the fourth harmonics of the roots of (13) with regard to the quadratic

$$pu_1^2 + qu_2^2 + ru_3^2 = 0 \quad \ldots\ldots\ldots(15),$$

the relation (14) is thus the homographic relation which was sought. Now let us return to the coordinates used in (1) and (2). For pu_1, qu_2, ru_3 we substitute $u'x-x'u$, $u'y-y'u$, $u'z-z'u$, respectively, and we have $p:q:r = x':y':z'$, where $x':y':z'$ are the coordinates of P, and $u = x+y+z$.

Thus the pair of lines (15) become

$$y'z'(u'x-x'u)^2 + z'x'(u'y-y'u)^2 + x'y'(u'z-z'u)^2 = 0,$$

or
$$x'y'z'u^2 - u'(y'z'x^2 + z'x'y^2 + x'y'z^2) = 0 \quad \ldots\ldots(16);$$

which, it is to be observed, represents the tangents drawn to the conic

$$y'z'x^2 + z'x'y^2 + x'y'z^2 = 0 \quad \ldots\ldots\ldots(17)$$

at the points where it is intersected by the line u.

Suppose now that we consider two points and seek the intersection of the diagonals of the quadrilateral formed by the two corresponding pairs of lines, we may put

$$\left.\begin{aligned} U_1 &= \frac{x^2}{x_1} + \frac{y^2}{y_1} + \frac{z^2}{z_1} - \frac{u^2}{u_1} \\ U_2 &= \frac{x^2}{x_2} + \frac{y^2}{y_2} + \frac{z^2}{z_2} - \frac{u^2}{u_2} \end{aligned}\right\} \quad \ldots\ldots\ldots(18),$$

then taking the discriminant of $U_2 + \lambda U_1$, we have

$$\frac{x_1x_2}{x_1+\lambda x_2} + \frac{y_1y_2}{y_1+\lambda y_2} + \frac{z_1z_2}{z_1+\lambda z_2} = \frac{u_1u_2}{u_1+\lambda u_2} \quad \ldots(19),$$

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which gives

$$\lambda\{u_2(x_2y_2z_1^2+z_2x_2y_1^2+y_2z_2x_1^2)-x_2y_2z_2u_1^2\}$$
$$+u_1(x_1y_1z_2^2+z_1x_1y_2^2+y_1z_1x_2^2)-x_1y_1z_1u_2^2=0\ldots(20),$$

and then $U_2+\lambda U_1$ gives the diagonals, the intersection of the diagonals being given by

$$x:y:z:u=\frac{x_1x_2}{x_1+\lambda x_2}:\frac{y_1y_2}{y_1+\lambda y_2}:\frac{z_1z_2}{z_1+\lambda z_2}:\frac{u_1u_2}{u_1+\lambda u_2}\ldots(21).$$

Now let us apply these results to the case in which the points are the circular points I, J. Let F_1, F_2 be a pair of points of intersection of the pair of lines (16) corresponding to I, with the pair of lines corresponding to J. Then taking circular coordinates X, Y, if

$$X^2-c^2=0,\quad Y^2-c^2=0\ldots\ldots\ldots\ldots\ldots(22)$$

are the equations of the lines IF_1, IF_2; JF_1, JF_2, respectively, the conditions that the rays $(X-x)(X-x')=0$ should be harmonically connected with $X^2-c^2=0$, and the rays $(Y-y)(Y-y')=0$ should be similarly connected with $Y^2-c^2=0$, are

$$xx'=c^2,\quad yy'=c^2\ldots\ldots\ldots\ldots\ldots(23),$$

so that the rays intersect in points connected by the inverse relations in polar coordinates

$$rr'=c^2,\quad \theta+\theta'=0\ldots\ldots\ldots\ldots\ldots(24),$$

where the origin is the middle point C, of F_1, F_2, or the intersection of the diagonals of the quadrilateral formed by the pairs of lines, such as (16) corresponding to I and J. It thus appears from (14) that the tangents from I, J to the cubics (7) and (8) are connected by the relations (23), that is, that the foci of the cubic (8) are the transformed points of the circular cubic (7) by the substitution (23) or (24). Now, to find the point C, we have from (21)

$$\alpha:\beta:\gamma:\delta=\frac{\alpha_1\alpha_2}{\alpha_1+\lambda\alpha_2}:\frac{\beta_1\beta_2}{\beta_1+\lambda\beta_2}:\frac{\gamma_1\gamma_2}{\gamma_1+\lambda\gamma_2}:\frac{\delta_1\delta_2}{\delta_1+\lambda\delta_2}\ldots(25),$$

where α_1, β_1, γ_1, δ_1; α_2, β_2, γ_2, δ_2 refer to the circular points. But $\alpha_1\alpha_2=\beta_1\beta_2=\gamma_1\gamma_2=\delta_1\delta_2$, so that from

$$\alpha:\beta:\gamma=\frac{1}{\alpha_1+\lambda\alpha_2}:\frac{1}{\beta_1+\lambda\beta_2}:\frac{1}{\gamma_1+\lambda\gamma_2},$$

we see that C lies on the circle round the triangle $\alpha\beta\gamma$.

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In the same way C lies on the circle round $\alpha\beta\delta$, and on the circles $\beta\gamma\delta$, $\gamma\alpha\delta$. Now we know that these four circles pass through a point, namely, the focus of the parabola touching the four lines α, β, γ, δ. This point is, then, the point C sought, namely, the origin of the transformation (24).

Again, suppose that the identical relation connecting α, β, γ, δ is

$$l\alpha + m\beta + n\gamma + p\delta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (26),$$

then the pair of lines, such as (16) passing through I, is

$$\frac{l\alpha^2}{\alpha_1} + \frac{m\beta^2}{\beta_1} + \frac{n\gamma^2}{\gamma_1} + \frac{p\delta^2}{\delta_1} = 0 \quad \ldots\ldots\ldots\ldots (27),$$

and the pair of lines through J is

$$\frac{l\alpha^2}{\alpha_2} + \frac{m\beta^2}{\beta_2} + \frac{n\gamma^2}{\gamma_2} + \frac{p\delta^2}{\delta_2} = 0 \quad \ldots\ldots\ldots\ldots\ldots (28),$$

or

$$l\alpha_2\alpha^2 + m\beta_2\beta^2 + n\gamma_2\gamma^2 + p\delta_2\delta^2 = 0 \quad \ldots\ldots\ldots (29),$$

$$l\alpha_1\alpha^2 + m\beta_1\beta^2 + n\gamma_1\gamma^2 + p\delta_1\delta^2 = 0 \quad \ldots\ldots\ldots (30),$$

respectively. That is, the pairs of lines (29) and (30) passing through I and J are the polar conics of J and I respectively, with regard to the cubic

$$l\alpha^3 + m\beta^3 + n\gamma^3 + p\delta^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots (31),$$

so that I, J must be corresponding points on the Hessian of (31), namely

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} + \frac{p}{\delta} = 0 \quad \ldots\ldots\ldots\ldots\ldots (32),$$

and it can easily be verified that this is the case, namely, that (32) is a circular cubic with its double focus on itself; it is, in fact, the locus of foci of conics touching the four lines α, β, γ, δ. The line F_1F_2 is thus one of the factors of the polar conic of the real point at infinity on (32) with regard to the cubic (31), and the point C is the corresponding point of the point just mentioned, that is, C is the double focus of (32). Also, the points F_1, F_2 are the foci of the Cayleyan of the cubic (31) which touches the line at infinity. To find the equation of the line F_1F_2 and the perpendicular to it at C, we take the origin at C and the axis of the parabola touching α, β, γ, δ as axis of x. Then the lines are

$$\Sigma \frac{l}{p_1}(x\cos\alpha + y\sin\alpha - p_1)^2 = 0 \quad \ldots\ldots\ldots (33),$$

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where $\Sigma l(x\cos\alpha + y\sin\alpha - p_1) = 0$, identically. Also we have

$$p_1\cos\alpha = p_2\cos\beta = p_3\cos\gamma = p_4\cos\delta,$$

since the origin is the focus of the parabola touching α, β, γ, δ. Thus (33) becomes

$$\Sigma\frac{l}{p_1}(x\cos\alpha + y\sin\alpha)^2 = 0,$$

or, finally,

$$(x^2 - y^2)\sin s - 2xy\cos s = 0 \quad \ldots\ldots\ldots\ldots(34),$$

where $s = \alpha + \beta + \gamma + \delta$, which is equal to 4δ, since the line of inflexions meets the curve at angles whose sum vanishes.

That is, F_1F_2 is one of the lines

$$x\cos 2\delta + y\sin 2\delta = 0, \quad x\sin 2\delta - y\cos 2\delta = 0 \ldots(35).$$

Again, F_1, F_2 lie on the conic

$$l\alpha^2\sin(\alpha - \delta) + m\beta^2\sin(\beta - \delta) + n\gamma^2\sin(\gamma - \delta) = 0 \ldots(36),$$

which reduces to

$$(x^2 - y^2)\cos 3\delta + 2xy\sin 3\delta + \frac{m^2}{\cos\alpha\cos\beta\cos\gamma} = 0 \ldots(37),$$

where $p_1 = \dfrac{m}{\cos\alpha}$ for the parabola touching α, β, γ, δ. But $m = 2R\cos\alpha\cos\beta\cos\gamma$, where R is the radius of the circle round the triangle α, β, γ. Hence if we substitute $x = \rho\sin 2\delta$, $y = \rho\cos 2\delta$ from (35), where $\rho = \frac{1}{2}F_1F_2$, in (37), we get

$$\rho^2 = -4R^2\frac{\cos\alpha\cos\beta\cos\gamma}{\cos\delta} \quad \ldots\ldots\ldots\ldots(38).$$

Hence, finally, it appears that, if the line through the three inflexions of a nodal cubic U meet the curve at angles whose sum vanishes, there is a circular cubic V having three inflexions and inflexional tangents in common with the given curve, and the transformed points of the foci of V by the substitution (24) are the foci of the nodal cubic U. The substitution (24) is determined by the origin being taken at the focus C of the parabola touching α, β, γ, δ, and the points F_1, F_2 at the two foci of the Cayleyan of the cubic (31), or the points determined by the equations (35) and (38). Or thus, taking the transformation of the circular cubic V by the substitution (24), we get a bicircular quartic W, and this bicircular quartic has all its foci in common with U. This bicircular quartic W evidently passes through C. Also, it is to be observed that

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the three points of inflexion and the vertices of the triangle formed by the inflexional tangents, namely, the points

$$\alpha\delta, \beta\gamma;\ \beta\delta, \gamma\alpha;\ \gamma\delta, \alpha\beta \quad \ldots\ldots\ldots\ldots\ldots (39)$$

are three pairs of corresponding points on the cubic (32). These pairs of points are, therefore, conjugate with respect to all the polar conics of (31), and therefore with respect to the polar conics of I, J, namely, the pairs of lines (29) and (30), or (22). From this it follows that the pair of lines (35) are the double lines of the involution determined by the lines drawn from C to the pairs of points (39), or are the tangents to the conics drawn through C to touch the lines α, β, γ, δ.

Thus the pairs of points (38) are connected by the relations (23) or (24), from which it follows that the vertices of the triangle formed by the inflexional tangents are on the quartic W, and, moreover, that the circles round the triangles $\beta\gamma\delta$, $\gamma\alpha\delta$, $\alpha\beta\delta$ are the osculating circles of the quartic at $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, respectively, which, thus, all meet the quartic again at C. Hence we see that if S_1, S_2, S_3, S_4 are the circles round the triangles $\beta\gamma\delta$, $\gamma\alpha\delta$, $\alpha\beta\delta$, $\alpha\beta\gamma$, respectively, and ρ_1, ρ_2, ρ_3, R are their radii, the equation

$$\frac{S_1 S_2 S_3}{\rho_1\rho_2\rho_3} - \frac{S_4^2}{R^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots (40),$$

after having been divided by the square of the distance from the point common to the four circles, gives the bicircular quartic W. The results with regard to the quartic W may be verified by means of elliptic integrals, namely, by means of the arguments of points on a bicircular quartic and Darboux's differential equation of confocal bicircular quartics. Now by a particular application of Abel's theorem, if

$$\phi,\ \psi \quad \ldots\ldots\ldots\ldots\ldots (41)$$

are two binary quartics, and

$$\phi + \lambda\psi = \text{a perfect square} \quad \ldots\ldots\ldots\ldots\ldots (42),$$

identically, then

$$\Sigma \pm \frac{(xdy - ydx)}{\sqrt{\phi}} = 0 \quad \ldots\ldots\ldots\ldots\ldots (43),$$

where the summation refers to the four roots of $\psi = 0$.

Then from (1) I observe that the tangential equation of the nodal cubic is

$$\alpha^{-\frac{1}{2}} + \beta^{-\frac{1}{2}} + \gamma^{-\frac{1}{2}} = 0,$$

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which, being cleared of radicals, can be written

$$\Sigma = \{\gamma(\alpha+\beta)-\alpha\beta\}^2 - 4\alpha\beta\gamma^2 = 0, \text{ \&c.} \ldots\ldots(44),$$

or $$\{2\gamma^2+\gamma(\alpha+\beta)-\alpha\beta\}^2 - 4\gamma^2(\alpha+\beta+\gamma) = 0 \ldots(45).$$

Now, if we combine the tangential equation with the equation of a point P,

$$x'\alpha + y'\beta + z'\gamma = 0,$$

we get the pencil of tangents drawn from P to the cubic. Hence, if the pencil of tangents coincide with ϕ, (41), and $\alpha\beta\gamma^2$, &c., coincide with ψ, we see from (42) and (43), and the equations (44) and (45), that

$$u_1+u_2-2u_3=0,\ u_3+u_1-2u_2=0,\ u_2+u_3-2u_1=0 \ldots(46),$$

$$3u_1 = u \ldots\ldots\ldots\ldots\ldots\ldots\ldots(47),$$

where u_1, u_2, u_3, u are the integrals such as (43) corresponding to the rays drawn from P to the points α, β, γ (yz, zx, xy) and the nodes $\alpha+\beta+\gamma=0$, or $x=y=z$, respectively. We thus get

$$u_2-u_3=\tfrac{1}{3}w,\quad u_3-u_1=\tfrac{1}{3}w,\quad u_1-u_2=-\tfrac{2}{3}w \ldots(48),$$

where w $(=2mK+2niK')$ is a complete period of u. Again, taking a point Q so that the anharmonic ratios of the pencils of tangents from P and Q to the curve are equal, we have

$$v_2-v_3=\tfrac{1}{3}w,\quad v_3-v_1=\tfrac{1}{3}w,\quad v_1-v_2=-\tfrac{2}{3}w \ldots(49).$$

Hence from (48) and (49) we get

$$u_1-v_1=u_2-v_2=u_3-v_3 \ldots\ldots\ldots\ldots(50).$$

Now let P, Q be the circular points, then Darboux has shown that the equation

$$u-v=\sigma \text{ (a constant)} \ldots\ldots\ldots\ldots(51)$$

represents a bicircular quartic touching the two pencils of tangents, that is, in the case we are considering, a bicircular quartic confocal with the nodal cubic (1). Hence, from (50), we see that a confocal bicircular quartic passes through the vertices A, B, C of the triangle formed by the inflexional tangents. Again, since for four concyclic points on the quartic there is $\Sigma' u =$ a constant, we see from (47) and (48) that the osculating circles at A, B, C meet the curve again at the same point. Also from (47) we see that the confocal quartic $u-v=3\sigma$ passes through the node of the cubic.

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In the preceding investigations we have obtained the equation of a bicircular quartic having its foci in common with the curve; thus these points may be considered as determined, as the foci of a bicircular quartic are supposed to be known. The method fails if the line passing through the inflexions is the line at infinity. In this case we see from (3) that S vanishes for the pencils of tangents from the circular points, so that the foci form a self-polar tetrad of points. Now I have remarked elsewhere that in such a case there are three distinct systems of confocal bicircular quartics. And, as for each system there is a pair of circular cubics, we have in the same case six circular cubics confocal with the given curve. I proceed to show how to obtain the equations of these cubics.

In this case the nodal cubic can be expressed by means of a parameter thus:

$$x = \frac{3b\theta^2 + 3c\theta}{1+\theta^3}, \quad y = \frac{3b'\theta^2 + 3c'\theta}{1+\theta^3} \ldots\ldots\ldots (52),$$

where the coordinates are circular, and the origin is at the node which coincides with the centre of gravity of the foci, the foci being given by

$$x^4 - 6bcx^2 - 4(b^3 + c^3)x - 3b^2c^2 = 0 \ldots\ldots\ldots (53),$$

and a similar equation in y. Now, writing the cubic

$$(lx + my)\,xy + ax^2 + by^2 + 2cxy + 2gx + 2fy + h = 0 \ldots (54),$$

I observe that it can easily be shown that the centre of gravity of the foci coincides with the centre of the polar conic of the real point at infinity on the curve. This gives

$$ax^2 + by^2 + 2cxy = k(lx + my)^2,$$

so that for the tangents to the curve from a circular point, we have, taking the discriminant of (54) with regard to x,

$$(my^2 + 2klmy + 2g)^2 - 4l(y + kl)(km^2y^2 + 2fy + h) = 0,$$

or $$m^2y^4 + 4(mg - 2lf)y^3 + 4\{2kl(mg - lf) - lh\}y + 4g^2 - 4kl^2h = 0 \ldots (55).$$

Similarly we have

$$l^2x^4 + 4(lf - 2mg)x^3 + 4\{2km(lf - mg) - mh\}x + 4f^2 - 4km^2h = 0 \ldots (56),$$

and making these equations coincide with (53), which we write

$$x^4 + 6\gamma x^2 + 4\delta x - 3\gamma^2 = 0, \quad y^4 + 6\gamma' y^2 + 4\delta' y - 3\gamma'^2 = 0,$$

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we get (54) in the form

$$2lm\,(lx+my)\,xy+\frac{(\delta l^2-\delta' m^2)}{\gamma l^2-\gamma' m^2}(lx+my)^2$$
$$-2l\,(2\gamma l^2+\gamma' m^2)\,x-2m\,(2\gamma' m^2+\gamma l^2)\,y-(\delta l^2+\delta' m^2)=0\ldots(57),$$

where $\frac{l}{m}$ is determined from $(\delta^2+4\gamma^2)\,l^6=(\delta'^2+4\gamma'^2)\,m^6$, or

$$l^2(b^2-c^2)\pm m^2(b'^2-c'^2)=0\ \ldots\ldots\ldots\ldots(58).$$

From this equation it follows that the asymptotes of the six cubics are parallel to the sides of a regular hexagon.

IV. *On the foci of some non-singular plane cubics.*

I propose to determine here some focal properties of certain non-singular cubics which satisfy special relations with the circular points at infinity. I commence by observing that the $m-3$ absolute invariants of the pencil of m tangents drawn from a point P to a plane curve are expressible in terms of the coordinates of P and the constant of the curve. Hence, eliminating the coordinates of P, we must have $m-5$ relations connecting the absolute invariants of the pencil with the constants of the curve. From this fact, it readily follows that if two points P, Q are connected with the curve by two properly chosen relations, then the pencils of tangents drawn from P, Q to the curve will have the same absolute invariants, or, in other words, be homographically connected. From this we see that the m points of intersection of the rays of one pencil with the corresponding rays of the other pencil lie on a conic passing through P, Q. Now let P, Q coincide with the circular points; we see, then, that if the curve satisfy two relations with the circular points at infinity all the foci will lie on a circle.

Now let us apply this result to the cubic. The absolute invariants of the pencil of six tangents drawn from a point to a cubic U are expressible in terms of the absolute invariant of the cubic and the functions U, H, Θ, where H is the Hessian and Θ a sextic covariant. Hence, referring the cubic to one of its canonical forms, we have

$$U=x^3+y^3+z^3+6mxyz\ \ldots\ldots\ldots\ldots(1),$$

and the pencils whose vertices are x, y, z; x', y', z' will be homographically connected, if we have

$$\frac{x^3+y^3+z^3}{xyz}=\frac{x'^3+y'^3+z'^3}{x'y'z'}\ldots\ldots\ldots\ldots(2),$$

$$\frac{x^3y^3+y^3z^3+z^3x^3}{(xyz)^2}=\frac{x'^3y'^3+y'^3z'^3+z'^3x'^3}{(x'y'z')^2}\ldots\ldots(3)$$

(see Salmon's *Higher Plane Curves*).

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It is to be oberved that the two relations are consistent with the pencils being homographically connected, but that it does not follow that the two relations are satisfied if, conversely, it is given that the pencils are homographic.

Now, from (2) and (3), we have

$$\frac{x^3}{x'^3} = \frac{y^3}{y'^3} = \frac{z^3}{z'^3} = \frac{xyz}{x'y'z'} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

or

$$\frac{x^3}{y'^3} = \frac{y^3}{x'^3} = \frac{z^3}{z'^3} = \frac{xyz}{x'y'z'},$$

or similar relations; or again

$$\frac{x^3}{y'^3} = \frac{y^3}{z'^3} = \frac{z^3}{x'^3} = \frac{xyz}{x'y'z'},$$

or the similar relation. But it is to be observed that the latter sets of relations are merely equivalent to the equations (4) for one of the three other canonical triangles. Now, from (4), we have

$$\frac{x}{x'} = \frac{\theta y}{y'} = \frac{\theta^2 z}{z'} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where θ is a cube root of unity, so that if we write the cubic

$$l\alpha^3 + m\beta^3 + n\gamma^3 + 6\mu\alpha\beta\gamma = 0 \quad \ldots\ldots\ldots\ldots(6)$$

in trilinear coordinates, the conditions (4) give for the two points $\frac{\alpha_1}{\alpha_2} = \frac{\theta\beta_1}{\beta_2} = \frac{\theta^2\gamma_1}{\gamma_2}$, and these equations for the circular points evidently express that the triangle of reference is equilateral. Thus, if the triangle of reference is equilateral, the pencils of tangents drawn to the cubic (6) from the circular points are homographically connected, and the six foci of the curve are situated on a circle. Further, the circle is the circumscribing circle of the triangle of reference; for from (5) we see that rays passing through a point I intersect corresponding rays passing through the corresponding point J on a conic circumscribing the triangle xyz and passing through I, J, *i.e.* the circumscribing circle of the triangle $\alpha\beta\gamma$. As a verification of this result it may be noticed that if the cubic (6) is circular the double focus, as well as the four single foci, should lie on the circumscribing circle. Now the cubic (6) is circular if

$$l + m + n + 6\mu = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(7),$$

in which case it is to be observed that the centre of the triangle

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of reference is on the curve. The tangents to the curve at the circular points are then easily found to be

$$\left.\begin{aligned}(2l-m-n)\,\alpha+(2m-n-l)\,\theta^2\beta+(2n-l-m)\,\theta\gamma=0\\(2l-m-n)\,\alpha+(2m-n-l)\,\theta\beta+(2n-l-m)\,\theta^2\gamma=0\end{aligned}\right\}\ldots(8),$$

which intersect in

$$\alpha:\beta:\gamma=(2l-m-n):(2m-n-l):(2n-l-m);$$

and this point lies on $\alpha\beta+\beta\gamma+\gamma\alpha=0$, viz. the circumscribing circle.

Again, by making two corresponding inflexional tangents pass through I, J respectively, we see that the inflexional focus and the four single foci of the cubic

$$U=l^2\alpha^3+m^3\beta^3+n^3\gamma^3+3lm\,(l+m)\,\alpha\beta\gamma=0\ \ldots.(9)$$

lie on the circle $\alpha\beta+\beta\gamma+\gamma\alpha=0$, the inflexional focus being $\frac{1}{l}, \frac{1}{m}, -\frac{1}{l+m}$.

Further, making $n=0$, we see that if the triangle formed by the nodal tangents and the line of inflexions of the nodal cubic

$$l\alpha^3+m\beta^3+6\mu\alpha\beta\gamma=0\ \ldots\ldots\ldots\ldots\ldots(10)$$

is equilateral, then the four foci lie on the circle circumscribing that triangle. On this, using rectangular coordinates, the four foci of the nodal cubic

$$l\,(x+y\sqrt{3})^3+m\,(x-y\sqrt{3})^3+n\,(x-3a)\,(x^2-3y^2)=0\ldots(11)$$

lie on the circle $x^2+y^2-4ax=0$.

I now proceed to mention two cases in which the pencils of tangents from I, J satisfy invariant relations, namely, form pencils in involution and break up into two sets of three which satisfy the combinantive relation $Q=0$ with each other respectively.

Now the pencil of tangents drawn to a cubic from a point on a harmonic polar is a system in involution; for the cubic can be projected so as to have the harmonic polar for an axis of symmetry, and it is evident that the tangents drawn from an axis of symmetry are in involution, the axis being one of the two double rays of the involution. Now the harmonic polars of

$$U=l\alpha^3+m\beta^3+n\gamma^3+6\mu\alpha\beta\gamma=0$$

are $$(l\alpha^3-m\beta^3)\,(m\beta^3-n\gamma^3)\,(n\gamma^3-l\alpha^3)=0\ldots\ldots(12),$$

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so that, making

$$\alpha \sqrt[3]{l} - \theta\beta \sqrt[3]{m} = 0, \quad \alpha \sqrt[3]{l} - \theta^2\beta \sqrt[3]{m} = 0 \ldots\ldots\ldots(13)$$

pass through the circular points, we get $l = m$, and the angle $C = 60°$. The doube rays of the two pencils evidently intersect in the vertex $\alpha\beta$ and the point

$$\alpha\sqrt{3} = 4\gamma \sin A, \quad \beta\sqrt{3} = 4\gamma \sin B, \text{ or } \alpha : \beta : \gamma = a : b : \tfrac{1}{2}c \ldots(14).$$

If k is half the distance between these points, and we take the origin at their middle point, the equations of the two pencils of tangents in circular coordinates will be of the form

$$\left.\begin{aligned}(x^2 + k^2 - 2\alpha_1 x)\,(x^2 + k^2 - 2\alpha_2 x)\,(x^2 + k^2 - 2\alpha_3 x) = 0 \\ (y^2 + k^2 - 2\beta_1 y)\,(y^2 + k^2 - 2\beta_2 y)\,(y^2 + k^2 - 2\beta_3 y) = 0\end{aligned}\right\} \ldots(15),$$

so that the foci form three pairs of points connected by the relations $xx' = yy' = k^2$, or, in polar coordinates, $rr' = k^2$, $\theta + \theta' = 0$.

It may be observed that from this result we see that if the cubic be transformed by the substitution

$$x + \frac{k^2}{x} = 2X, \quad y + \frac{k^2}{y} = 2Y \ldots\ldots\ldots(16),$$

the transformed curve will have three foci determined by

$$(X - \alpha_1)\,(X - \alpha_2)\,(X - \alpha_3) = 0, \quad (Y - \beta_1)\,(Y - \beta_2)\,(Y - \beta_3) = 0 \ldots\ldots\ldots(17).$$

If we make I, J lie on the same harmonic polar, that is, if the line at infinity is a harmonic polar, the curve has a centre and may be written

$$(a, b, c, d)\,(x, y)^3 + gx + fy = 0 \ldots\ldots\ldots(18),$$

and in this case it is evident that the foci are determined by equations of the form

$$x^6 - p_1 x^4 + p_2 x^2 - p_3 = 0, \quad y^6 + lc = 0 \ldots\ldots\ldots(19),$$

that is, that the foci lie by fours on three parallelograms about the origin as centre. Hence, if we transform this curve (18) by the substitution $x^2 = X$, $y^2 = Y$, we get a nodal cubic with a focus at the origin on the curve and three foci given by

$$X^3 - p_1 X^2 + p_2 X - p_3 = 0, \text{ \&c.}$$

I have shown elsewhere that the locus of the vertices of

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pencils in involution circumscribed about a cubic consists of twelve cubics in addition to the nine harmonic polars. Hence, if one of these cubics is circular, and the origin be taken at the middle point of the pair of points determined by the intersection of the double rays of the two pencils, the foci will be determined by equations similar to (15).

I now consider the case in which the pencils of tangents break up into sets of three satisfying the combinantive relation $Q=0$ with each other. I observe that if a cubic be written in the form

$$x^3+y^3+z^3+6mxyz=0,$$

its tangential equation can be written

$$\{\lambda^3+\mu^3-(1+16m^3)\,\nu^3-12m^2\lambda\mu\nu\}^2 - 4\,(1+8m^3)\,(\lambda\mu+2m\nu^2)^3=0 \quad\ldots\ldots(20),$$

where the curves within the bracket are a curve of the third class and a conic, respectively, touching six inflexional tangents of the curve. From this equation it follows that if we draw tangents to the curve from any point P on the conic

$$\lambda\mu+2m\nu^2=0 \quad\ldots\ldots(21)$$

they will break up into two pencils of three lines each of the form

$$ABC\pm D^3=0 \quad\ldots\ldots(22),$$

where ABC are the tangents drawn from P to the curve of the third class and D is the tangent at P to the conic. Now let tangents in this case be drawn from the circular points, that is, let $\lambda\mu+2m\nu^2=0$ be a circle, then the cubic can be written in trilinear coordinates

$$U=l\alpha^3+m\beta^3+n\,(4\gamma^3-3\alpha\beta\gamma)=0 \quad\ldots\ldots(23),$$

and the angles of the triangle are such that $A=B$.

For this cubic then we have two curves of the third class, each having three foci in common with the cubic, whose tangential equations are

$$\phi_3\pm k\rho^3=0 \quad\ldots\ldots(24),$$

where ϕ_3 is the curve of the third class within the brackets in (20) and ρ is the centre of the circle (21), that is, taking the origin at the latter point, the foci of the cubic (23) are given in circular coordinates by

$$\left.\begin{aligned} x^3-p_1x^2+p_2x-p_3\pm kx^3=0\\ y^3-p'_1x^2+p'_2x-p'_3\pm ky^3=0\end{aligned}\right\} \quad\ldots\ldots(25),$$

where $x^3-p_1x^2+p_2x-p_3=0$, y^3- &c. $=0$ give the foci of ϕ_3.

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In the case of the nodal cubic there is a modification of this latter result. Suppose the cubic Σ'_1 to be given in tangential coordinates as the envelope of the line

$$3\lambda\theta^2 + 3\mu\theta + \nu(1+\theta^3) = 0 \quad \ldots\ldots\ldots\ldots(26),$$

then by the theory of binary cubics we have, identically,

$$G^2 - 4H^3 = P^2\Sigma \quad \ldots\ldots\ldots\ldots\ldots\ldots(27),$$

where G is a curve of the third class, H is the conic

$$(\lambda^2 - \nu\rho)\,k^2 + (\nu^2 - \lambda\mu)\,k + \mu^2 - \lambda\rho \quad \ldots\ldots(28),$$

Σ is the tangential equation of the cubic, and P is the point

$$3\lambda k^2 + 3\mu k + \nu(1+k^3) \quad \ldots\ldots\ldots\ldots\ldots(29)$$

on Σ. Now H is evidently a conic touching the curve and also the three inflexional tangents

$$\lambda = \mu^2\nu \text{ and } \lambda:\mu:\nu = \theta:\theta^2:1\,;\ \ \lambda:\mu:\nu = \theta^2:\theta:1,$$

so that if the cubic is written

$$x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0 \quad \ldots\ldots\ldots\ldots\ldots(30),$$

the conic H is of the form

$$\left(\frac{x}{l}\right)^{\frac{1}{3}} + \left(\frac{y}{m}\right)^{\frac{1}{3}} + \left(\frac{z}{n}\right)^{\frac{1}{3}} = 0 \quad \ldots\ldots\ldots\ldots(31),$$

where $$l + m + n = 0.$$

Now let H be a circle, then considering lines drawn from the circular points we see from (27) that three foci of Σ will be given by

$$G + 2\rho^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(32),$$

while one focus and the square of the point P will belong to

$$G - 2\rho^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(33),$$

where ρ is the centre of H.

Now using trilinear coordinates we see that in this case the cubic

$$(l\alpha)^{\frac{1}{3}} + (m\beta)^{\frac{1}{3}} + (n\gamma)^{\frac{1}{3}} = 0 \ldots\ldots\ldots\ldots(34)$$

must touch the inscribed circle of the triangle, that is, must satisfy the relation

$$\frac{l}{\cos^2\frac{1}{2}A} + \frac{m}{\cos^2\frac{1}{2}B} + \frac{n}{\cos^2\frac{1}{2}C} = 0 \ldots\ldots\ldots(35).$$

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It can then be shown that the fourth focus given by (33) lies on the circumscribing circle; for, reciprocating the equation of

$$\cos\tfrac{1}{2}A\sqrt{(\beta\gamma)} + \cos\tfrac{1}{2}B\sqrt{(\gamma\alpha)} + \cos\tfrac{1}{2}C\sqrt{(\alpha\beta)} = 0 \ldots (36)$$

which represents a tricuspidal quartic passing through the circular points (see Salmon's *Conics*, Art, 311) and identifying the result with the tangential equation of (34) we have, since the origin is a focus,

$$\frac{\sin\theta_1 \sin^2\tfrac{1}{2}\theta_1}{l\alpha'} = \frac{\sin\theta_2 \sin^2\tfrac{1}{2}\theta_2}{m\beta'} = \frac{\sin\theta_3 \sin^2\tfrac{1}{2}\theta_3}{n\gamma'} \ldots (37),$$

where θ_1, θ_2, θ_3 are the angles the sides of the triangle subtend at a focus α', β', γ'. Now these equations (37), subject to the condition (35), are satisfied for the point

$$mn\sin A\cos^2\tfrac{1}{2}A,\ nl\sin B\cos^2\tfrac{1}{2}B,\ lm\sin C\cos^2\tfrac{1}{2}C \ldots (38)$$

on the circumscribing circle.

I now proceed to mention a case in which we can have several curves of the same degree with all their foci in common. Using areal coordinates x, y, z, if a curve U can be expressed as a symmetric function of x, y, z, we can find five other curves having all their foci in common with U. In the same case it is evident that the tangential equation of U is a symmetric function of λ, μ, ν, where λ, μ, ν are the perpendiculars from the vertices of the triangle of reference on a line. To find the foci then, we substitute for λ, μ, ν, $x-a_1$, $x-a_2$, $x-a_3$, respectively, in circular coordinates, obtaining the equation

$$f(x-a_1, x-a_2, x-a_3) = 0 \ldots\ldots\ldots\ldots (39),$$

and, in the same way for the variable y,

$$f(y-b_1, y-b_2, y-b_3) = 0 \ldots\ldots\ldots\ldots (40),$$

where $a_1, b_1 : a_2, b_2 : a_3, b_3$ are the coordinates of the vertices of the triangle of reference. Now these equations (39) and (40) are by hypothesis symmetric functions of a_1, a_2, a_3 and b_1, b_2, b_3 respectively. We see thus that the equations determining the foci are precisely the same for all the other five curves corresponding to the five triangles of reference, the coordinates of whose vertices are

$$a_1b_1, a_2b_2, a_3b_3;\ a_1b_2, a_2b_3, a_3b_1;$$

$$a_1b_3, a_2b_1, a_3b_2;\ a_1b_2, a_2b_3, a_3b_1;\ a_1b_3, a_2b_1, a_3b_2,$$

respectively.

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In the case of the cubic we have

$$U=a(x+y+z)^3+b(x+y+z)(xy+yz+zx)+cxyz..(41),$$

and for this curve we can show that the line at infinity contains three inflexions: for putting

$$x=\alpha+\beta+\gamma,\ y=\alpha+\theta\beta+\theta^2\gamma,\ z=\alpha+\theta^2\beta+\theta\gamma,$$

U becomes

$$27a^3+9ba(\alpha^2-\beta\gamma)+c(\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma)..(42),$$

from which it follows that α or the line at infinity contains three inflexions. We can thus obtain six cubics, each of which has three inflexions at infinity, having all their foci in common.

Again, if the equation of a curve be symmetrical in two of the variables, x, y, say, in areal coordinates, the equations giving the foci will be symmetrical in a_1, a_2 : b_1, b_2, respectively, so that the curves corresponding to the two triangles

$$a_1b_1,\ a_2b_2,\ a_3b_3;\ \ a_1b_2,\ a_2b_1,\ a_3b_3$$

are confocal, that is, the curves corresponding to the triangles A, B, C; A', B', C', where A', B' are the antipoints of A, B, are confocal.

The cubic in this case is of the form

$$U=az^3+bz^2(x+y)+z\{c(x+y)^2+dxy\}$$
$$+(x+y)\{e(x+y)^2+fxy\}=0\ \ ...(43),$$

and this, by putting $x=\alpha+\beta$, $y=\alpha-\beta$, can be shown to have an inflexion at infinity. We can thus have two confocal cubics, each with an inflexion at infinity, a result which I have proved in a different manner elsewhere.

With regard to curves of the fourth order it may be noticed *en passant* that we can have six confocal curves whose equations in oblique Cartesian coordinates are of the form

$$x^2y^2+ax^3+by^3+cxy+d=0\ \ .....(44),$$

and that we can have two confocal curves of the form

$$y^4+y^2(\alpha,\beta,\gamma)(x,1)^2+(a,b,c,d,e)(x,1)^4=0..(45).$$

I now proceed to investigate a case in which when a cubic U is made subject to a single condition, we can find another cubic having its foci in common with U. If a cubic have a circle osculating it in two points, it must evidently be subject to a single condition: and such a curve can be written

$$AS-B^2=0\ .....(46),$$

where S is a circle and A and B are lines.

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Now I shall generate this cubic as the envelope of a system of conics having double contact with the circle, and taking the locus of the foci of the system I shall obtain another cubic confocal with the given cubic. Using circular coordinates, the envelope of the system of conics

$$\theta^2 (ax + by + c) + \theta (lx + my + n) = \sqrt{(xy - r^2)} \ldots (47),$$

where θ is a parameter, is evidently of the form (46).

Now the two foci of the conic

$$(\alpha x + \beta y + \gamma)^2 = xy - r^2 \ldots\ldots\ldots\ldots\ldots (48)$$

which are collinear with the centre of the circle $xy = r^2$ are given by

$$\frac{x}{\beta} = \frac{y}{\alpha} = t,$$

$$t^2 (1 - 4\alpha\beta) - 4\gamma t - 4r^2 = 0 \ldots\ldots\ldots\ldots (49),$$

Hence for a pair of foci of (47) we have

$$\frac{x}{b\theta^2 + m} = \frac{y}{a\theta^2 + l} = \theta t,$$

$$t^2 \{1 - 4\theta^2 (a\theta^2 + l) (b\theta^2 + m)\} - 4\theta (c\theta^2 + n) t - 4r^2 = 0,$$

whence, eliminating θ, we get

$$(ax - by)^2 - 4 (ma - lb) (lx - my) \Sigma = 0 \ldots\ldots (50),$$

where Σ is the circle

$$(ma - lb) (xy + r^2) + n (ax - by) - c (lx - my).$$

The confocal cubic (50) is thus a curve of precisely the same kind as (46), having the circle Σ as a double osculating circle.

Further, it is to be remarked that the lines A', B' for the cubic (50) are the perpendiculars from the centre of S on A, B, and that the latter lines are the perpendiculars on A', B' from the centre of Σ; also that S and Σ cut orthogonally. Hence, we see that the confocal cubics (46) and (50) are reciprocally related.

The simplest non-singular cubic in relation to the circular points is the cubic with an inflexional focus and the line at infinity as an inflexional tangent. This curve has two single foci collinear with the inflexional focus. Taking rectangular coordinates the curve may be written

$$(x - \alpha)^3 - \mu \{y^2 + (x - \beta)^2\} = 0,$$

where β, 0 is the inflexional focus. Expressing now that $\pm c$, 0 are the single foci, we get

$$\mu = \tfrac{27}{16} (\beta - 2\alpha), \quad c^2 = (\beta - 2\alpha) (2\alpha - 3\beta),$$

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so that if we assume

$$\alpha = \tfrac{1}{4}c\left(3t + \frac{1}{t}\right), \quad \beta = \tfrac{1}{2}c\left(t + \frac{1}{t}\right),$$

we have $\mu = -\frac{27}{16}ct$, and the equation of the system of curves with their two single foci in common is

$$\{4tx - c(3t^2+1)\}^2 + 27ct^2\{4t^2y^2 + (2tx - ct^2 - c)^2\} = 0.$$

From this equation it can be shown that five curves of the system pass through a point.

It may be observed that this curve is such that being given four concyclic points the directrix ($x = \alpha$) touches a tricuspidal quartic U with the line at infinity as bitangent, and that the tangential Hessian of U, namely a tricuspidal quartic of the same kind, touches all the lines joining the four points and the asymptotes of all the hyperbolæ passing through the four points.

V. *On curves confocal with a plane cubic.*

I propose to determine here certain curves confocal with a cubic, the means of investigation that I employ being the theorem that the locus of the foci of a singly infinite system of curves is, in general, confocal with the envelope of the system. I observe that a cubic is determined by nine conditions and six foci, so that the six foci are connected by three relations. Hence the confocal curves must be subjected to the conditions that their six foci, or six of their foci, must satisfy these three conditions. I consider exclusively the case in which the generating curves of the cubic are conics, and I consider almost entirely the case in which the direct equations of the conics involve a parameter rationally in the second degree. As an exception to the latter I consider some cases in which the variable system of conics has double contact with a fixed conic S, when by taking U in the form $\sqrt{S}$ the variable conic is expressed in such a way as to involve a parameter in the second degree.

Now, if U is a cubic and H is its Hessian, the polar conic of a line with regard to U is a conic having triple contact with H. Hence, if we take lines passing through a fixed point P, we have a system of conics having triple contact with H and touching the polar line L of P with regard to U. There are twelve such systems of conics corresponding to the

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four poles of L with regard to U and the three cubics such as U which have a given Hessian.

If we put

$$U = x^3 + y^3 + z^3 + 6mxyz \ldots\ldots\ldots\ldots(1),$$

we have $$H = m^2(x^3 + y^3 + z^3) - (1 + 2m^3)\,xyz \ldots\ldots\ldots(2),$$

$$G = m(\alpha^3 + \beta^3 + \gamma^3) + (1 - 4m^3)\,\alpha\beta\gamma \ldots\ldots\ldots(3),$$

and the polar conic of the line $L = \alpha x + \beta y + \gamma z = 0$ is

$$V = \alpha^2(yz - m^2x^2) + \beta^2(zx - m^2y^2) + \gamma^2(xy - m^2z^2)$$
$$+ 2\beta\gamma(m^2yz - mx^2) + 2\gamma\alpha(m^2zx - my^2) + 2\alpha\beta(m^2xy - mz^2)$$
$$\ldots\ldots(4).$$

It is to be observed that the discriminant of V is the square of the Cayleyan G. We see thus that when L passes through a fixed point, V belongs to a system of which two may be drawn through an arbitrary point.

Now, for the foci of a conic whose equation in circular coordinates is

$$(a, b, c, f, g, h)(x, y, 1)^2 = 0 \ldots\ldots\ldots\ldots(5),$$

we have $$Cx^2 - 2Gx + A = 0, \quad Cy^2 - 2Fy + B = 0 \ldots\ldots(6),$$

where A, B, &c., are the tangential coefficients.

Hence, if L is the line at infinity, V is a parabola, and, as I have shown elsewhere, we thus obtain a circular unicursal quintic confocal with H, there being twelve such confocal quintics corresponding to a given curve H. It may be observed that the general circular unicursal quintic involving, as it does, twelve constants, is the simplest curve with six unrelated foci.

Now, when L is an arbitrary line, we have from (6)

$$x = \frac{G \pm \Theta\sqrt{b}}{C}, \quad y = \frac{F \pm \Theta\sqrt{a}}{C} \ldots\ldots\ldots(7),$$

where $D = -\Theta^2$, Θ being an expression of the third degree in a parameter θ, and a, b, c involve θ in the second degree, so that A, B, &c., are of the fourth degree in θ. Now a, b evidently correspond to the pairs of conics of the system which pass through the circular points respectively. Also, since V is the envelope of the polar lines with regard to U of all points on the line L, it follows that C, being the condition that V should touch the line at infinity, is the product of the tangential equations of the four poles of the line at infinity, P_1, P_2, P_3, P_4,

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say, each of which in the case under consideration is a linear function of the parameter, that is, when L passes through a fixed point. Similarly, A, B are the products of the tangential equations of the four poles of x, y, namely, the lines joining an arbitrary point K to I, J respectively. Let these products be $Q_1Q_2Q_3Q_4$, $R_1R_2R_3R_4$ respectively, then we have

$$G^2 - \Theta^2 V_1 = P_1P_2P_3P_4Q_1Q_2Q_3Q_4 \ldots\ldots\ldots(8),$$

$$F^2 - \Theta^2 V_2 = P_1P_2P_3P_4R_1R_2R_3R_4 \ldots\ldots\ldots(9),$$

and G, F are the conditions that KI, KJ should be conjugate with regard to V. Now let us consider the general cubic, and let L be so determined that a is proportional to b in (5) the generating conic. This makes two circles, given by $a = b = 0$, belong to the system, and then the axes of (5) are parallel to given lines. It is easy to see then that the point P is the intersection of two common tangents of the polar conics with regard to U of the circular points, that is, is one of six points related to U in a certain manner. Suppose

$$U = \alpha x^3 + \beta y^3 + \gamma z^3 + \delta u^3 \ldots\ldots\ldots\ldots(10),$$

where $x + y + z + u = 0$, identically, then

$$H = \beta\gamma\delta yzu + \gamma\delta\alpha zux + \delta\alpha\beta uxy + \alpha\beta\gamma xyz \ldots.(11),$$

and then, if we take u as the line joining

$$I, J, (x_1, y_1, z_1)(x_2, y_2, z_2),$$

the polar conics of I, J are

$$\alpha x_1 x^2 + \beta y_1 y^2 + \gamma z_1 z^2 = 0, \ \alpha x_2 x^2 + \beta y_2 y^2 + \gamma z_2 z^2 = 0 \ldots(12),$$

or, in tangential coordinates,

$$\frac{\lambda^2}{\alpha x_1} + \frac{\mu^2}{\beta y_1} + \frac{\nu^2}{\gamma z_1} = 0, \quad \frac{\lambda^2}{\alpha x_2} + \frac{\mu^2}{\beta y_2} + \frac{\nu^2}{\gamma z_2} = 0 \ldots(13),$$

so that the common tangents are

$$x\sqrt{(\alpha x_1 x_2)} \pm y\sqrt{(\beta y_1 y_2)} \pm z\sqrt{(\gamma z_1 z_2)} = 0 \ldots\ldots(14).$$

Hence L, being the polar with regard to U of the intersection of two common tangents, say a pair intersecting on z, is

$$\alpha\beta(xy_1y_2 + yx_1x_2) + \delta u\{\sqrt{(\alpha x_1 x_2)} \pm \sqrt{(\beta y_1 y_2)}\}^2 = 0 \ldots(15),$$

so that, given U, there are six such lines. Hence, corresponding to three cubics such as U having a given Hessian, we have 18 different systems of conics of the kind we have been investigating.

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Now for the foci we have, when $a=b$,

$$x=\frac{G\pm\Theta\sqrt{a}}{P_1P_2P_3P_4},\quad y=\frac{F\pm\Theta\sqrt{a}}{P_1P_2P_3P_4}\ldots\ldots\ldots(16).$$

If a is of the form $(\theta-\alpha)(\theta-\beta)$, these expressions can be rationalised by the substitution $\theta=\frac{\alpha-\beta t^2}{1-t^2}$.

Observing, then, that since $G^2-\Theta^2 a$ is divisible by $P_1P_2P_3P_4$, which is of the eighth degree in t, we see that the rational expression $G\pm\Theta\sqrt{a}$ must be divisible by an expression in t of the fourth degree. We thus have

$$x=\frac{f_4}{\phi_2\psi_2},\quad y=\frac{g_4}{\phi_2\chi_2}\ldots\ldots\ldots\ldots\ldots(17),$$

$$x=\frac{f_4'}{\chi_2\omega_2},\quad y=\frac{g_4'}{\psi_2\omega_2}\ldots\ldots\ldots\ldots\ldots(18),$$

where f_4, g_4, f_4', g_4' are of the fourth degree in t, and ϕ_2, ψ_2, χ_2, ω_2 are of the second degree. These curves, being unicursal sextics with I, J as double points, have six foci, and the latter points, as we see by the investigation, coincide with the foci of the cubic.

The two sextics are thus confocal, and from either of them we can generate an entire singly infinite system of such sextics. If t_1, t_2 are the roots of $\phi_2=0$, and we put $\frac{t-t_1}{t-t_2}=\theta$, we get from (17)

$$x=\frac{f_4}{\theta(\alpha\theta^2+\beta\theta+\gamma)},\quad y=\frac{g_4}{\theta(\alpha'\theta^2+\beta'\theta+\gamma')}\ldots(19),$$

where f_4, g_4 are expressions of the fourth degree in θ. But this curve arises from making $\phi=\theta$ in

$$x=\frac{f_4(\theta)}{\theta(\alpha\theta^2+\beta\theta+\gamma)},\quad y=\frac{g_4(\phi)}{\phi(\alpha'\phi^2+\beta'\phi+\gamma')}\ldots(20),$$

so that, taking $\phi=\lambda\theta$, $\phi\theta=\mu$, we obtain orthogonal systems of unicursal sextics with I, J as double points, having their foci in common with the given cubic.

We can now see how we can have unicursal quartics confocal with the cubic in a certain special case. Suppose in (20) that we have

$$\frac{\beta^2}{\alpha\gamma}=\frac{\beta'^2}{\alpha'\gamma'}\ldots\ldots\ldots\ldots\ldots(21),$$

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then, corresponding to the values $\phi = \frac{\beta'\alpha}{\alpha'\beta}\theta$,

$$\phi\theta = \frac{\gamma'\beta}{\gamma\beta'} \ldots\ldots\ldots\ldots\ldots\ldots (22),$$

it is evident that the quadratics $\alpha\theta^2 + \beta\theta + \gamma$, $\alpha'\phi^2 + \beta'\phi + \gamma'$ coincide, so that, if the cubic is subject to the condition (22), we have a pair of orthogonal unicursal quartics confocal with the cubic. In the case (22), it may be observed, the anharmonic ratio of ϕ, ψ_1 in (17) is equal to the anharmonic ratio of ϕ, χ_1.

I shall now show otherwise that in a certain case, that is, if the cubic satisfy a special relation, we can have a unicursal quartic confocal with the given curve. Taking rectangular axes parallel to the given direction of the axes of the variable conic in the case under consideration, its equation may be written

$$ax^2 + by^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots\ldots (23),$$

where a, b, &c., are expressions in θ of the second degree.

Now, if this conic touch a fixed line L parallel to an axis, the enveloping cubic must satisfy a special condition; thus, since the origin is arbitrary, we take $L = y = 0$, and then $g^2 = ac$, which is satisfied by $a = 1$, $c = \theta^2$, $g = \theta$, so that (23) becomes

$$x^2 + (a\theta^2 + b\theta + c)y^2 + 2\theta x + 2(a'\theta^2 + b'\theta + c')y + \theta^2 = 0 \ldots (24),$$

the enveloping cubic of which is

$$(ay^2 + 2a'y + 1)(cy + 2c') + (ay + 2a')x^2 = y(by + b')^2 + 2x(by + b') \ldots (25),$$

whose asymptotes, being parallel to $y = 0$, $ax^2 + (ac - b^2)y^2 = 0$, are such that one bisects the angle between the other two, or the triangle formed by the asymptotes is isosceles. In this case one of the loci of the foci is a unicursal quartic, being determined by

$$x + \theta = 0, \quad y^2(a\theta^2 + b\theta + c) + 2y(a'\theta^2 + b'\theta + c') + (a'\theta^2 + b'\theta + c')^2 = 0 \ldots . (26),$$

which has a node at infinity. A cubic, thus, whose asymptotes form an isosceles triangle, has its foci in common with a trinodal quartic with a node at infinity. As a special case consider a system of conics touching the given cubic H at a given point. The line L is then the tangent at that

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point, and the point P is the corresponding point on H, for we know that the polar of a point on H with regard to U is the tangent to H at the corresponding point.

Now, taking circular coordinates and the origin at the fixed point on H, we write the variable conic

$$ax^2+by^2+2hxy+2g(x+y)=0 \quad \ldots\ldots\ldots(27),$$

and since $\Delta, =(2h-a-b)g^4$, is to be a perfect square, we have $2h=a+b-t^2$, say. We have then for the foci

$$\left.\begin{aligned} x&=\frac{2g}{(t+\sqrt{b})^2-a}\\ y&=\frac{2g}{(t+\sqrt{a})^2-b}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(28).$$

This gives the inverse of the curve

$$x:y:1=\{(t+\sqrt{b})^2-a\}:\{(t+\sqrt{a})^2-b\}:2g \ldots(29).$$

Now if the given cubic H is circular, a and b are perfect squares, and we get the foci as the inverse points of the foci of the pair of conics represented by (12). These conics, it may be observed, both pass through the origin. Again, let b be proportional to a, in which case a pair of chords of intersection of the polar conics of I, J with regard to U are mutually rectangular, that is, let the cubic satisfy this particular condition, then from (28) we see that the foci are the inverse points with regard to a node of the six foci of the pair of unicursal quartics represented by (29).

I now proceed to show that in the general case in which L passes through an arbitrary point we can generate from the locus by certain transformations curves confocal with the cubic. If b is proportional to $(\theta-\theta_1)(\theta-\theta_2)$, then, putting

$$\theta=\frac{\theta_1-\theta_2 t^2}{1-t^2} \quad \ldots\ldots\ldots\ldots(30),$$

$\sqrt{b}$ is rational in t.

Similarly if a is proportional to $(\theta-\theta'_1)(\theta-\theta'_2)$, putting

$$\theta=\frac{\theta'_1-\theta'_2 t'^2}{1-t'^2} \quad \ldots\ldots\ldots\ldots(31),$$

$\sqrt{a}$ is rational in t'.

We thus obtain for the foci two pairs of equations, each pair of which is of the form

$$x=\frac{f_4(t)}{u_4(t)}, \quad y=\frac{g_4(t')}{v_4(t')} \quad \ldots\ldots\ldots\ldots(32),$$

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where f_4, u_4 are rational expressions in t of the fourth degree, and g_4, v_4 are rational expressions in t' of the same degree. Now it is evident that we shall obtain confocal curves if we assume any other relation connecting t, t' instead of the one resulting from (30) and (31), as the foci depend upon the discriminants of the equations (32) in t, t', respectively, taken separately. Hence if we take a homographic relation

$$Att' + Bt + Ct' + D = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (33),$$

say, connecting t, t', we evidently obtain unicursal curves confocal with the given cubic, no matter what values A, B, C, D have.

If A, B, C, D are arbitrary, the curve represented by (32) is a unicursal curve of the eighth degree with I, J as quadruple points. Again, if A, B, C, D are subjected to the two conditions which make two linear factors of U_4 coincident with two linear factors of V_4, we get singly infinite systems of sextics with I, J as double points. Now if we express that a root of a biquadratic is connected with a root of another biquadratic by a homographic relation (33), it is evident that the resulting condition is one of 16 different conditions, so that there are $\frac{16.15}{2} = 120$ pairs of such conditions. It thus seems that we have 120 different systems of confocal and orthogonal unicursal sextics, or 240 systems, if we consider the pair of curves such as (32).

Again, let A, B, C, D be determined so as to make three factors of U_4 coincide with three factors of V_4, then the curve (32) is a circular unicursal quintic. We have thus, apparently, $\frac{16.15.14}{1.2.3} = 560$ such quintics, or corresponding to the pair (32), 1120 such quintics, in all.

Further, if U_4, V_4 have the same anharmonic ratio, U_4 can be transformed into V_4 by a homographic relation, subject to which (32) represents a unicursal quartic.

I now consider a system of conics having triple contact with a cubic H in the case in which they are the polar conics with regard to U of lines touching a given conic Σ. The coordinates α, β, γ of the line L are then quadratic expressions in a parameter, so that the conic $V(4)$ involves the parameter in the fourth degree.

Hence if we take the discriminant of V with regard to the parameter we get an equation of the twelfth degree, but this, as we shall verify, is divisible by the square of H, as three of

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the points of contact with the envelope lie on H, the remaining factor, which is of the sixth degree, giving the locus of the fourth point of contact. Let

$$\Sigma = (A, B, C, F, G, H)(\alpha, \beta, \gamma)^2 \ldots \ldots\ldots(34)$$

be the conic touched by L, and let P, Q, R be the three differential coefficients of Σ, then for the envelope of V we have

$$\frac{\frac{dV}{d\alpha}}{P} = \frac{\frac{dV}{d\beta}}{Q} = \frac{\frac{dV}{d\gamma}}{R} \ldots\ldots\ldots\ldots\ldots\ldots(35).$$

Now, recollecting that V considered as a function of α, β, γ is the tangential equation of the polar conic of x, y, z, and that the discriminant of the polar conic is H, we get

$$A\Sigma - P^2 = (AB - H^2)\beta^2 + (AC - G^2)\gamma^2 + 2\beta\gamma(AF - GH)$$
$$= H\{z\beta^2 + y\gamma^2 - 2mx\beta\gamma),$$

so that we have, after dividing by H,

$$\left.\begin{aligned} y\gamma^2 + z\beta^2 - 2mx\beta\gamma &= kP^2 \\ z\alpha^2 + x\gamma^2 - 2my\gamma\alpha &= kQ^2 \\ x\beta^2 + y\alpha^2 - 2mz\alpha\beta &= kR^2 \end{aligned}\right\} \ldots\ldots\ldots\ldots(36),$$

so that, solving for x, y, z, observing that

$$P\alpha + Q\beta + R\gamma = 0,$$

and neglecting the factor $\alpha\beta\gamma$, we obtain

$$\left.\begin{aligned} \rho x &= m(R^2\beta + Q^2\gamma) + (2m^2P^2 - QR)\alpha \\ \rho y &= m(P^2\gamma + R^2\alpha) + (2m^2Q^2 - PR)\beta \\ \rho z &= m(Q^2\alpha + P^2\beta) + (2m^2R^2 - PQ)\alpha \end{aligned}\right\} \ldots(37),$$

which represents a unicursal curve, W, of the sixth degree, that is, the locus of the fourth point of contact is a unicursal curve of the sixth order.

Now, when we seek the locus of the foci of this system, let us consider what conditions Σ should be subject to in order that a simplification of the locus should take place. Since the discriminant is a perfect square, namely, the square of the equation giving the six common tangents of Σ and the Cayleyan, the expressions for x and y in (7) will be rational, if the conditions that V should pass through the circular points are perfect squares. In this case Σ has double contact with the polar conics of the circular points with regard to U. There are thus nine such systems corresponding to the three

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systems of conics having double contact with two fixed conics, and the three cubics such as U with a given Hessian; and each of these systems involves an arbitrary parameter, as a conic having double contact with two conics contains an undetermined constant. We have then for the locus of the foci of one of these systems

$$x=\frac{f_8\pm\Theta_6 f_2}{\phi_2\psi_2\chi_2\omega_2},\quad y=\frac{g_8\pm\Theta_6 g_2}{\phi_2\psi_2\chi_2\omega_2} \qquad \text{.....(38),}$$

where f_8, g_8 are expressions of the eighth degree in the parameters, f_2, g_2 are expressions of the second degree, and ϕ_2, &c. are quadratic expressions which correspond to the points of contact of the tangents drawn to Σ from the four points of intersection of the polar conics of the circular points, namely, the four poles with regard to U of the line at infinity.

Now $(f_8+\Theta_6 f_2)(f_8-\Theta_6 f_2)$ is to be divisible by $\phi_2\psi_2\chi_2\omega_2$; hence $f_8+\Theta_6 f_2$ is divisible by $\phi_2\psi_2$ and $f_8-\Theta_6 f_2$ by $\chi_2\omega_2$, and the case is similar for $g_8\pm\Theta_6 g_2$. Hence from (38) we have

$$x=\frac{f_4}{\chi_2\omega_2},\quad y=\frac{g_4}{\chi_2\phi_2} \qquad \text{.....(39),}$$

and

$$x=\frac{f_4'}{\phi_2\psi_2},\quad y=\frac{g_4'}{\psi_2\omega_2} \qquad \text{..... .....(40);}$$

that is, a pair of unicursal sextics. It is evident that the pairs of factors $\phi_2\psi_2$, $\chi_2\omega_2$ of $\phi_2\psi_2\chi_2\omega_2$ correspond to one of the three systems of conics having double contact with the polar conics of I, J, the entire three pairs of factors being derived from the three systems. Hence we have pairs of unicursal sextics, confocal with a given non-singular cubic, both belonging to one of nine different systems, each of which involves an arbitrary parameter, which agrees with what we have already proved concerning confocal unicursal sextics with I, J as double points.

In this case since the conditions that the generating conic V should pass through the circular points I, J, respectively, are perfect squares, it follows that I, J are double points of the envelope. Hence, as I, J are not on the cubic, they must be double points on the unicursal sextic (37).

Now I have shown elsewhere that four polar conics can be described to have double contact with a given conic Σ, from which it appears that two other double points of the sextic are the two points which give rise to the two other polar conics having double contact with Σ.

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This sextic, then, being unicursal and having I, J as double points, has six foci, and since it is an envelope of the conics V it must be confocal with the locus of the foci, namely, the curves (32), that is, the sextic is confocal with the given cubic H.

Let us now apply the preceding results to the circular cubic. In this case it is to be observed that the coordinates of the foci of V are expressible as rational functions of α, β, γ, the coordinates of the line L; for then the tangential equations of the polar conics of I, J are respectively the squares of the equations, δ, δ', say, of the corresponding points of I, J. Since then $(G+\Theta\delta)(G-\Theta\delta)$ is divisible by $P_1P_2P_3P_4$, where P_1, &c. are the tangential equations of the four poles of the line at infinity with regard to U, we have

$$x=\frac{Q_1Q_2}{P_1P_2},\quad y=\frac{R_1R_2}{P_1P_2} \qquad \text{.....(41),}$$

and

$$x=\frac{Q_3Q_4}{P_3P_4},\quad y=\frac{R_3R_4}{P_3P_4} \qquad \text{.....(42),}$$

where Q_1, Q_2, Q_3, Q_4 and R_1, R_2, R_3, R_4 are the tangential equations of the poles of x, y respectively with regard to U, since the condition that V should touch a line is the product of the equations of the four poles of the line. It is to be observed that the points Q_1, Q_2 are collinear with the pair of points P_1P_2, the line being one of the factors of the polar conic of I.

Similarly, R_1R_2 are collinear with P_1P_2: the line being one of the factors of the polar conic of J, and in the same way there are similar relations for the points involved in (42).

Hence we may write

$$Q_1Q_2=lP_1^2+mP_2^2+nP_1P_2,$$
$$R_1R_2=l'P_1^2+m'P_2^2+n'P_1P_2,$$

so that, if we put

$$P_1=P_2\theta,\quad P_1=P_2\phi \qquad \text{.....(43),}$$

we have, from (41),

$$x=\frac{l\theta^2+n\theta+m}{\theta},\quad y=\frac{l'\phi^2+n'\phi+m'}{\phi} \qquad \text{.....(44).}$$

This gives the coordinates of the foci of a conic having triple contact with a circular cubic, independently of any other relation the conic may satisfy. Corresponding, then, to any

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relation between ϕ and θ we have a curve given by (43) touching L, and a locus of one focus of V, given by (44), having in all cases two foci in common with the cubic, the locus of the other focus, given by (42), having the two other foci of the cubic as its foci. As for the other pair of foci or antipoints of the foci just mentioned we have

$$\left.\begin{aligned} x &= \frac{Q_1Q_2}{P_1P_2}, \quad y = \frac{R_2R_4}{P_2P_4} \\ x &= \frac{Q_3Q_4}{P_3P_4}, \quad y = \frac{R_1R_3}{P_1P_3} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(45).$$

It thus appears that in general there is a separate locus for each of the four foci, real and imaginary, of conics having triple contact with a circular cubic and satisfying another condition.

Now let us take $\phi = \lambda\theta$ in (45), when we get $P_3 = \lambda P_2$, that is, the line L passes through a fixed point on the line joining two poles of the line at infinity, and (44) give a conic having two foci in common with the curve. Also, since, if we put $P_4 = P_3\theta' = P_2\phi'$, we have, in the same case, $\theta' = \lambda\phi'$, it follows from (42) that the locus of the other focus given by (44) is another conic having for its foci the two other foci of the cubic. We thus have two conics, which conjointly have their four foci in common with the cubic. Similarly, if L pass through a fixed point on the line joining the poles P_1, P_4 of the line at infinity, we get from (44) another pair of conics which conjointly have their four foci in common with the cubic.

It may be observed that such a pair of conics intersect on the cubic H; for, considering the special cases when L coincides with one of the tangents drawn from the fixed point to the Cayleyan, the polar conic V breaks up into a pair of tangents intersecting at a point O of the Hessian. We thus get two points on H, for we omit the point on H arising from the line joining P_2P_3, which is a tangent to the Cayleyan, as it is irrelevant. These two points are evidently at the intersection of the loci of the foci. This agrees with what is already known; for (see Salmon's *Higher Plane Curves*, Art. 279), the cubic can be generated as the locus of points whose distances ρ_1, ρ_2, ρ_3, ρ_4 from four fixed concyclic points F_1, F_2, F_3, F_4 are connected by the relation

$$\frac{\rho_1 + \rho_3}{F_1F_3} = \frac{\rho_2 + \rho_4}{F_2F_4} \quad \ldots\ldots\ldots\ldots(46),$$

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which shows that conics having F_1, F_2 as foci intersect conics having F_3, F_4 as foci on the curve, if they have equal eccentricities.

Now, if we consider the three cubics such as U of which H is the Hessian, we get three pairs of conics, each pair having their four foci in common with the curve. These three pairs correspond to the three ways in which four points can be grouped in two pairs. The process, thus, that we have made use of, constitutes the algebraic solution of the problem: given a circular cubic to find the foci, the solution of the biquadratic involved being made to depend upon finding one of the three cubics of which the given cubic is the Hessian.

We can also get a conic having two foci in common with the cubic by taking $\phi\theta =$ a constant in (44). In this case the line L touches the conic whose tangential equation is $P_1^2 = kP_3P_4$, which passes through P_3, P_4 and touches P_1P_2, P_1P_3. But in this case we have no other conic with the other two foci of the cubic as foci.

In the general case in which L passes through an arbitrary fixed point, we see from the preceding expressions for the coordinates of the foci of V that the locus of the foci consists of four circular unicursal cubics, which may be divided into two pairs, members of one pair having the foci F_1, F_2 of the cubic as foci, and of the other pair having F_3, F_4 as foci.

When the line L touches an arbitrary conic, since the points I, J are double points on the envelope, and I, J are on the cubic, it follows that the unicursal sextic (37) touched by V passes through I, J and therefore has 8 foci. In the same case we see that the locus of the foci consists of four unicursal sextics with I, J as double points. Each of these sextics has two foci of the cubic as bitangent foci, and has two single foci elsewhere. We see hence that the 8 single foci of these four sextics coincide with the 8 foci of the circular unicursal sextic (37).

It may be observed that if the conic touched by L touches the sides of the triangle formed by three poles of the line at infinity, the locus of one of the foci is a circular nodal cubic with its foci at two foci of the given curve.

I now proceed to consider the case of the nodal cubic. In this case there is only one cubic U corresponding to a given cubic H. When the conic V, having triple contact with H touches a fixed line z, its equation can be written

$$V = \theta^2x^2 + 2\theta(xy + zu) + y^2 + zv = 0 \quad \ldots\ldots(47),$$

where

$$H = zu^2 + 2uxy - x^2v = 0 \quad \ldots\ldots\ldots\ldots(48),$$

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ux being the node. Since, then, for $\theta = \infty$, one conic of the system (47) is a perfect square, namely, x^2, the line joining the node to one of the points where z meets H, the tangential coefficient of V must be divisible by this value of θ, and thus be of the third degree in θ. And, in general, when $L = \alpha x + \beta y + \gamma z = 0$ is arbitrary, the tangential coefficients, being divisible by the equation of the node, are of the third degree in α, β, γ, and the discriminant is proportional to the square of the node multiplied by the square of the Cayleyan, which is a conic in this case. Hence, when z is the line at infinity and V is a parabola, the locus of the focus is a circular unicursal quartic, which thus has its four foci in common with the given curve. Again, if two conics of the system are circles, and the principal axes are parallel to given lines, the locus of the foci consists of unicursal circular quartics having their four foci in common with the given curve.

Again, if the line touches a conic Σ having double contact with the polar conics of I, J with regard to U, the locus of the foci consists of circular unicursal quartics confocal with the curve. Since Σ involves an arbitrary parameter, we thus have singly infinite systems of confocal circular unicursal quartics. I have shown this elsewhere as follows. The equations

$$\left.\begin{aligned} x &= \frac{a\theta^3 + b\theta^2 + c\theta + d}{\theta(\alpha + \beta\theta)} \\ y &= \frac{a'\phi^3 + b'\phi^2 + c'\phi + d'}{\phi(\alpha' + \beta'\phi)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (49)$$

in circular coordinates evidently give, if we take $\phi = \lambda\theta$, $\phi\theta = \mu$, confocal and orthogonal systems of circular unicursal quartics, unicursal cubics of the systems being given by $\alpha\beta'\lambda = \alpha'\beta$, $\alpha\alpha' = \beta\beta'\mu$.

I now propose to consider a different system of conics related to a nodal cubic. If a conic V have triple contact with a nodal cubic

$$x^3 + y^3 = xyz \quad \ldots\ldots\ldots\ldots\ldots\ldots (50),$$

then the points of contact are given by an equation of the form

$$\theta^3 - a\theta^2 + b\theta - 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots (51),$$

where $x = \theta$, $y = \theta^2$, $z = 1 + \theta^3$ for a point on the curve, that is, $\theta_1\theta_2\theta_3 = 1$, the relation for collinear points being $\theta_1\theta_2\theta_3 = -1$.

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Hence a conic W having double contact with V meets the cubic in two triads of points given by

$$l\theta + m\theta^3 + n(1+\theta^3) \pm p(\theta^3 - a\theta^2 + b\theta - 1) = 0 \ldots (52).$$

Considering, then, one triad, we can describe through three points θ_1, θ_2, θ_3 on the curve a definite conic W, namely,

$$W = (lx + my + nz)^2 = p^2 V = 0 \quad \ldots\ldots\ldots (53),$$

where l, m, n, p are linear expressions in the sum, sum of the products in pairs, and product of θ_1, θ_2, θ_3, p in particular being the condition that θ_1, θ_2, θ_3 should be collinear. Hence in the tangential equation of W the coefficients, being divisible by p^2, are functions of the second degree in the sum, &c., of θ_1, θ_2, θ. Also the discriminant Δ is proportional to p^6 multiplied by

$$(b\theta_1\theta_2 - \theta_1 - \theta_2 + a)(b\theta_2\theta_3 - \theta_2 - \theta_3 + a)(b\theta_3\theta_1 - \theta_1 - \theta_3 + a) \quad \ldots\ldots (54),$$

each of the factors being the condition that the line joining a pair of the points should touch V.

Now let $\theta_1 = \theta_2$, and let $\theta_3 = k$ be a fixed point on the curve, then we have a system of conics having double contact with V, passing through a fixed point on the curve and touching the curve once elsewhere.

Now consider the case in which V is a circle. Since V is the polar conic of a line with regard to the cubic having the given curve for its Hessian, it follows that the line must be a common tangent of the polar conics of I, J; that is, there are four circles having triple contact with the curve. In this case the conditions that W should pass through I, J respectively are perfect squares, and the only factor of Δ that is not a square is from (54), $b\theta^2 - 2\theta + a = \psi_2$, say. We have, then, for the circular coordinates of the foci of W

$$x = \frac{f_4 + f_2\phi_1\sqrt{\psi_2}}{h_4}, \quad y = \frac{g_4 + g_2\phi_1\sqrt{\psi_2}}{h_4} \quad \ldots\ldots (55),$$

Rationalising $\sqrt{\psi_2}$, then, by the substitution $\theta = \frac{\alpha - \beta t^2}{1 - t^2}$, where α, β are the roots of $\psi_2 = 0$, and observing that the numerators and denominators of the fractions must have a common factor of the fourth degree in t we get two unicursal quartics. These quartics are the locus of the foci of W collinear with the centre of V, the locus of the foci not collinear with the centre of V being unicursal curves of the

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eighth degree. The quartics have four foci at the foci of the given cubic and a bitangent focus at the fixed point $\theta = k$. This result may be considered otherwise. Suppose we have, in circular coordinates, the system of conics

$$ax + by + c + d\sqrt{(xy - k^2)} = 0 \quad \ldots\ldots\ldots(56),$$

namely, a system of conics having double contact with the circle $xy - k^2 = 0$, a, b, c, d being quadratic expressions in a parameter θ. The envelope of the system is a binodal quartic having quartic contact with a circle, that is, satisfying a certain condition, and so involving $12 - 1 = 11$ constants.

For the locus of the foci of this conic, which are collinear with the origin, we have $x : y : 1$ proportional to

$$b : a : (c + \Delta) \quad \ldots\ldots\ldots\ldots\ldots\ldots(57),$$

where $\Delta^2 = c^2 + k^2(d^2 - 4ab)$; that is, the locus is a binodal quartic, and this curve must have its eight foci in common with the envelope of the system.

Now there is evidently a linear relation connecting a, b, c, d, $la + mb + nc + pd = 0$, say, so that (56) will always pass through a fixed point if $l : m : n : p = x : y : 1 : \sqrt{(xy - k^2)}$ or $lm - p^2 = k^2 n^2$. Again, if Δ as a function of the parameter θ has a square factor, the locus of the foci is unicursal, as Δ then can be rationalised. Subject to these two conditions the envelope is divisible by a line passing through the fixed point and touching the circle $xy = k^2$, the remaining factor being a unicursal cubic not satisfying any particular condition. And this agrees with our previous investigation, that is, we have found the equation of a unicursal quartic with four given foci and with a bitangent focus elsewhere. Now a general unicursal quartic involves 11 constants, so that a unicursal quartic with a bitangent focus involves 9 constants, and this is the number of constants involved in a unicursal cubic and an arbitrary point thereof.

Making the point $\theta = k$ on the unicursal cubic variable we get the singly infinite system of unicursal quartics with four given single foci and a variable bitangent focus. We see thus, then, that the locus of the bitangent focus of the confocal system is a confocal unicursal cubic. Again, I observe that k occurs in the first degree in the expressions for the coordinates of a point on the quartic, so that for a given point, eliminating the parameter, we get an equation of the eighth degree in k. That is, eight quartics of the confocal system can be described to pass through a given point.

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To prove the existence of the bitangent focus belonging to the quartic as given by (57), I observe that in such a case we should have $k^2(d^2-4ab)+c^2-(c-\mu a)^2$ = a perfect square for some value of μ. Hence, putting $d=la+mb+nc$, and making

$$k^2(d^2-4ab)+2\mu ac-\mu^2a^2=k^2(mb+nc+\rho a)^2,$$

we get

$$\mu^2=k^2(l^2-\rho^2),\ \mu=k^2n(\rho-l),\ lm-2=m\rho,$$

whence eliminating ρ and μ we obtain $lm=1+k^2n^2$, and this relation is by hypothesis satisfied, as it expresses the condition that the conic W (56) passes through a fixed point.

As a property then of this confocal system of quartics, I observe that since we have $x:y=b:a$, it follows that a pair of bitangents of the curve intersect at the origin, namely, the centre of the fixed circle V.

Since there are four circles such as V having triple contact with a given nodal cubic, it follows that we have thus found four systems of unicursal quartics with a bitangent focus and four given single foci.

We might also consider the case in which the three points θ_1, θ_2, θ_3 on the cubic, through which W passes, coincide. W then osculates the cubic at the point θ, and l, m, n, p are expressions of the third degree in θ, and Δ is proportional to, from (54), $(b\theta^2-2\theta+a)^2$. Hence, if α, β are the roots of $b\theta^2-2\theta+a=0$, and $\theta-\alpha=t^2(\theta-\beta)$, we get, for the foci of W in this case, expressions of the form

$$x:y:1=(at^6+bt^4+ct^2+d):(a't^6+b't^4+c't^2+d')$$
$$:(a''t^6+b''t^4+kt^2+c''t^2+d'')\ldots(58),$$

that is, the locus is a unicursal curve of the sixth degree and eighth class. This curve must have four foci in common with the given cubic, but does not seem of sufficient interest to merit any further investigation.

We may notice the modification of the preceding results in the case in which the curve is a cuspidal cubic. The conic having triple contact with the curve is then replaced by a conic passing through the cusp and having double contact with the curve. If we put $x:y:1=1:\theta:\theta^3$ for the cubic we have

$$a^2yz+(b^2+2ac)y^2+c^2x^2+2abzx+2bcxy=V=0\ldots.(59)$$

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as the equation of such a conic, the points of contact being given by $a\theta^2+b\theta+c=0$. The tangential equation of V involves a, b, c in the third degree and the discriminant is proportional to the square of $a(2b^2+ac)$. If $\alpha a+\beta b+\gamma c=0$ the conic V touches also the line

$$2\alpha\beta x-(\beta^2+2\gamma\alpha)\,y+\gamma^2 z=0 \quad \ldots\ldots\ldots(60),$$

and there is no difficulty in finding the loci of the foci of V in the same way as for the nodal cubic.

By reciprocation then we can have the system of conics Σ touching the inflexional tangent and having double contact with the curve. Again, we can have a system of conics having double contact with V, touching the curve and passing through a fixed point thereof, as the conic W,

$$W=lx+my+nz+p\sqrt{V}=0 \quad \ldots\ldots\ldots\ldots(61)$$

meets the cubic in three points given by

$$l+m\theta+n\theta^2\pm p\,(a\theta^2+b\theta+c)=0\ldots\ldots\ldots(62),$$

the roots of which can be made to be θ, θ, k. And it is to be observed that there are four circles such as V.

Further, using tangential coordinates, we can have a system of conics having double contact with Σ, touching the curve and a fixed tangent thereof, by reciprocating the equation of W (61). And since the direct equation of the conic Σ is

$$a^3x^2+4aby^2-8c^3z^2+2c\,(ac-2b^2)\,yz$$
$$+2b\,(2b^2+3ac)\,zx-2a^2bxy=0 \quad \ldots\ldots\ldots(63),$$

it follows that there are nine circles belonging to the system Σ, as we find nine values of $a:b:c$ by substituting the coordinates of I, J in (63). This result may be investigated thus: let λ, μ, ν be tangential circular coordinates, then

$$a\lambda+b\mu+c\nu+d\sqrt{(\nu^2-k^2\lambda\mu)}=0 \quad \ldots\ldots\ldots(64)$$

where a, b, c, d are quadratic functions of a parameter, represents a conic having double contact with the circle $\nu^2-k^2\lambda\mu=\Sigma$, and also having double contact with a curve of the fourth class, with two bitangents, which has quartic contact with the circle Σ.

Now precisely as in the direct case this curve of the fourth class breaks up, subject to two conditions, into a tricuspidal quartic and a point F on Σ, and if a further condition is

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satisfied, the tricuspidal quartic becomes a cuspidal cubic and Σ becomes a circle having double contact with the cubic and passing through the cusp thereof. Now the foci of (64) are given by

$$x:y:1=\alpha:b:(c\pm d) \quad \ldots\ldots\ldots\ldots\ldots(65),$$

as we see by putting $\lambda\mu=0$. Hence for the locus of the foci we have two conics, and the foci of these conics must coincide with the foci of the curve of the fourth class. And in the particular cases of the tricuspidal quartic and cuspidal cubic, one conic (63) has two foci in common with the curve, and the other conic has one focus at the third focus of the curve and the other focus at the point F on Σ.

As a particular case of the curve of the fourth class, I may notice the nodo-bicuspidal quartic; that is, if such a quartic has quartic contact with a circle, we can find two conics having their foci in common with the curve. This is also the case if there be a circle touching the two inflexional tangents and having double contact with the curve. Thus if the curve is

$$(l\mu\nu+m\nu\lambda+n\lambda\mu)^2-k^2\lambda\mu\nu^2=0 \quad \ldots\ldots\ldots(66),$$

then if a conic of the system

$$k^2\nu^2+2\theta(l\mu\nu+m\nu\lambda+n\lambda\mu)+\lambda\mu=0 \quad \ldots\ldots(67)$$

is a circle, it is readily seen that we can at once find a pair of conics having their foci in common with the curve. For the nodal cubic there will be similar relations in the case of conics touching three inflexional tangents and the curve, and of conics having triple contact with the curve and touching one inflexional tangent. Further, we can have a system of conics having double contact with such a conic Σ and having double contact with the curve; for using tangential coordinates, we have, for the common tangents of

$$(l\lambda+m\mu+n\nu)^2-p^2\Sigma=0 \quad \ldots\ldots\ldots\ldots(68),$$

$$lf_1+mf_2+nf_3\pm pf=0 \quad \ldots\ldots\ldots\ldots(69),$$

where $\Sigma=f^2$,

This, being equated to the square of $\alpha\theta^2+\beta\theta+\gamma$, gives a relation of the second degree between α, β, γ, and thus makes l, m, n, p proportional to rational expressions of the fourth degree in a parameter t.

I propose now to consider another system of conics related to a general cubic, namely, conics described through two

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fixed points P, Q on the curve to have double contact with the curve elsewhere.

By projecting the curve so as to make P, Q coincide with the circular points, these conics become the generating circles of a circular cubic. We see thus that there are four distinct systems of such conics, corresponding to the four systems of generating circles of a circular cubic. As a simple case then we can take P, Q as two points at infinity on the curve.

The generating conic in this case can be written, in rectangular Cartesian coordinates,

$$ax^2 + by^2 + 2gx + 2fy + c = 0 \quad \ldots\ldots\ldots\ldots (70),$$

where a, b are constants and g, f, c are expressions of the second degree in a parameter. I have shown elsewhere that the locus of the foci of this conic consists of two confocal binodal quartics with the line at infinity as a bitangent.

Corresponding, thus, to the three pairs of points at infinity, there are twelve pairs of such confocal binodal quartics.

I now consider other cases. I observe that the discriminant of the conic V is proportional to $f_1^2\Theta_4$, where f_1 gives the case when V is divisible by the line PQ, and Θ_4 gives four pairs of corresponding tangents drawn to the cubic from P, Q respectively. Hence for the circular coordinates of the foci in this case, we have

$$x = \frac{G + f_1\sqrt{\Theta_4}\sqrt{b}}{C}, \quad y = \frac{F + f_1\sqrt{\Theta_4}\sqrt{a}}{C} \quad \ldots\ldots (71),$$

where a, b are the result of substituting the coordinates of I, J in V, respectively. Now suppose a proportional to b, that is, suppose two circles to belong to the system, or in other words that two conics can be described through P, Q, I, J to have double contact with the curve. Hence if we project the curve into a circular cubic, P, Q become a pair of points mutually inverse with regard to one of the Jacobian circles of the cubic, so that for the general curve P, Q are the pair of points where the equilateral hyperbola passing through the points of contact of tangents parallel to an asymptote meets the curve again. Also, it may be observed, P, Q are collinear with the finite point where the asymptote meets the curve. There are, thus, three pairs of points such as P, Q, so that there are altogether twelve systems of conics such as V. We have, then, from (71)

$$x = \frac{G + f_1\sqrt{\Theta_4}}{C}, \quad y = \frac{F + f_1\sqrt{\Theta_4}}{C} \quad \ldots\ldots\ldots (72),$$

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where

$$\Theta_6 = a\Theta_4, \quad G^2 - f_1^2\Theta_6 = AC, \quad F^2 - f_1^2\Theta_6 = BC,$$

and G, F, C, A, B are of the fourth degree in θ.

We see thus that the locus of the foci consists of a pair of sextic curves, each having 8 double points, among which I, J are included. Of these sextics the points P, Q are bitangent foci and the six foci of the cubic are single foci.

In this case if we take rectangular coordinates and the origin at the middle point of P, Q $(\pm\alpha, \pm\beta)$, V may be written

$$V = a(x^2 - \alpha^2) + b(y^2 - \beta^2) + 2c(\beta x - \alpha y) \quad \ldots(73),$$

where a, b, c are quadratic expressions in the parameter, and $a\alpha^2 + b\beta^2$ is a perfect square $= f_1^2$. Θ_6 then equals

$$(a - b)(c^2 + ab).$$

If $c^2 + ab$, and therefore Θ_6, has a square factor, the given cubic has a node, and the two sextic loci have nine nodes, two bitangent foci at P, Q, and four single foci at the foci of the given cubic.

Suppose the cubic to be circular and the points P, Q to be arbitrary points on the curve, then, for the foci of V, we have from (72)

$$x = \frac{G + f_1\phi_1\sqrt{\Theta_4}}{C}, \quad y = \frac{F + f_1\psi_1\sqrt{\Theta_4}}{C} \quad \ldots\ldots(74),$$

putting $b = \phi_1^2$, $a = \psi_1^2$. Hence we find that the locus of the foci consists of a pair of sextic curves with 9 nodes, among which I, J are included. These sextics have bitangent foci at P, Q and single foci at the four foci of the given curve.

In the case of the nodal cubic we may let P, Q coincide at the node of the curve, and then the system of conics having double contact with the curve is doubly infinite.

Let the cubic be given by

$$x : y : z = \theta : \theta^3 : (1 + \theta^2) \quad \ldots\ldots\ldots\ldots(75),$$

then a conic meeting the curve in the points given by

$$\theta(a\theta^2 + b\theta + c)^2 = 0,$$

is

$$V = (2bc - a^2)x^2 + (2ab - c^2)y^2 + (b^2 + 2ac)xy + a^2yz + c^2zx = 0$$

$$\ldots\ldots\ldots\ldots(76).$$

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The tangential coefficients of this conic are of the fourth degree in a, b, c, and the discriminant is $(a^2+c^2-abc)^2$.

In this case the locus of the foci has four foci at the foci of the curve, and a bitangent focus at the node of the curve. If we have a linear relation connecting a, b, c and we determine the constants in it so that V_1 is proportional to V_2, where V_1, V_2 are the result substituting the coordinates of I, J in V, the loci are unicursal sextics with I, J as double points.

I now propose to show how we can obtain binodal quartics, having six foci in common with a given non-singular cubic. A binodal quartic involves 12 constants and has eight foci, so that six of its foci can be arbitrarily assigned.

Now, since a cubic involves 9 constants its six foci are connected by three relations, so that the binodal quartic is subjected to three conditions in making six of its foci coincide with the foci of a cubic.

I commence by investigating circles having double contact with a cubic at two points P, Q which are collinear with a point of inflexion. Let xy be a point of inflexion O and z the corresponding harmonic polar, then the cubic can be written

$$z^2(fx+gy)-(ax^3+bx^2y+cxy^2+dy^3)=0 \quad ...(77),$$

from which we see that any conic of the form

$$\alpha x^2+\beta xy+\gamma y^2+\delta z^2=0 \quad .....(78)$$

meets the cubic in three pairs of points, such that each pair is collinear with O. Now let (78) be a circle having double contact with the curve, that is, let (78) pass through two points I, J, and be such that

$$(\alpha x^2+\beta xy+\gamma y^2)(fx+gy)+\delta(ax^3+bx^2y+cxy^2+dy^3)=0 \quad .....(79)$$

has a square factor. We thus get two linear relations and one relation of the fourth degree connecting α, β, γ, δ, that is, we determine four circles having double contact with the cubic at a pair of points collinear with a given point of inflexion O. Hence corresponding to the nine points of inflexion we determine 36 such circles.

Let S be one of those circles; let B be its chord of contact and A the line joining the points where it meets the cubic again, then, if the inflexional tangent is $lA+mB=0$, the cubic can be written

$$S(lA+mB)-AB^2=0 \quad .....(80).$$

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Now consider conics having double contact with S at points collinear with $O(AB)$, namely

$$V=\lambda A+\mu B+\nu\sqrt{S}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(81),$$

then these conics evidently are of the form (78) and so meet the cubic in three pairs of points collinear with AB.

Thus if a relation be established between λ, μ, ν, we have a system of conics having double contact with the cubic. If we put $l\alpha+m\beta=nat^2$, we may take

$$\lambda:\mu:\nu=(t^2+1)^2:(t^2-1):kt^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(82),$$

Now taking the origin at the centre of S and circular coordinates, V is of the form

$$\lambda(ax+by+c)+\mu(a'x+b'y+c')+\nu\sqrt{\{(xy-k^2)\}}=0\ldots(83).$$

Then for the foci of V collinear with the origin, we have

$$x:y:1=(\lambda b+\mu b'):(\lambda a+\mu a'):(\lambda c+\mu c'+\Delta)\ldots(84),$$

where $\Delta^2=(\lambda c+\mu c')^2+k^2\{\nu^2-4(\lambda a+\mu a')(\lambda b+\mu b')\}$.

Hence since the square only of the variable t is involved in these expressions, we see that the locus of the pair of foci referred to consists of a binodal quartic.

Now if we take the envelope of the tangent line to a unicursal cubic, we get the equation of the curve multiplied by the equation of the three inflexional tangents, so precisely in taking the envelope of (83) we get the equation of the cubic multiplied by the inflexional tangent $lA+mB$ and that particular conic V', say, of the system which doubly osculates the given cubic. We thus infer that the binodal quartic (82) has six foci at the foci of the given cubic and two foci at the foci of the conic V'.

If C, D represent a pair of lines passing through the centre of S, the binodal quartic can be represented thus

$$C:D:\sqrt{\Sigma}=(t^2+1)^2:(t^2-1):t^2 \quad \ldots\ldots\ldots(85),$$

where Σ is a circle orthogonal to S. The quartic then is such that it can be projected into a curve with an axis of symmetry, that is, it has a pair of bitangents intersecting on the line of the nodes. Again, it has a double osculating circle, namely, the circle Σ.

We may notice a particular case of the preceding. A special relation will be satisfied by the preceding if the cubic

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has a focus situated on the harmonic polar of a point of inflexion. If the radius of the circle S vanishes, its centre becomes a focus F on the corresponding harmonic polar, and then all the generating conics have a focus at F. For the locus of the other focus of the generating conics, we have then

$$x:y:1=4(\lambda b+\mu b')(\lambda c+\mu c'):4(\lambda a+\mu a')(\lambda c+\mu c')$$
$$:\{\nu^2-4(\lambda a+\mu a')(\lambda b+\mu b')\}\ldots\ldots\ldots\ldots(86),$$

which represents a unicursal quartic with a tacnode at F.

This unicursal quartic then has five of its foci at the five other foci of the cubic, and its sixth focus at the other focus of a particular generating conic of the system, namely, that one which doubly osculates the cubic.

As another special case we may notice that of the given cubic having a node, Δ then in (84) has a square factor and the locus (84) is unicursal, having four foci in common with the given curve, and two foci elsewhere. There are 12 circles such as S in this case.

VI. *On certain curves confocal with trinodal and binodal quartics.*

I now proceed to consider the methods of obtaining curves confocal with a unicursal quartic, and I consider first the different methods of generating the curve as the envelope of a system of conics involving a parameter in the second degree. I have shown elsewhere that there are four systems of conics having quartic contact with a trinodal quartic. Suppose the curve to be expressed in the form $x:y:z:f_1:f_2:f_3$, where f_1, f_2, f_3 are rational expressions in a parameter of the fourth degree, then, if $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ are the pairs of parameters of the three nodes respectively, we have, for four collinear points,

$$\phi(\alpha)-k_1\phi(\alpha')=0,\ \phi(\beta)-k_2\phi(\beta')=0,\ \phi(\gamma)-k_3\phi(\gamma')=0\ldots(1),$$

where $\phi(t)=(t-\theta_1)(t-\theta_2)(t-\theta_3)(t-\theta_4)$,

k_1, k_2, k_3 are constants, and

$$l\{\phi(\alpha)-k_1\phi(\alpha')\}+m\{\phi(\beta)-k_2\phi(\beta')\}+n\{\phi(\gamma)-k_3\phi(\gamma')\}=0$$
$$\ldots\ldots(2),$$

l, m, n being constants, is an identical relation.

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In the same way we find for eight points on a conic

$$\phi(\alpha) - k_1^2\phi(\alpha') = 0, \text{ \&c. } \ldots\ldots\ldots\ldots\ldots(3),$$

where $\phi(t) = (t-\theta_1)(t-\theta_2)\ldots(t-\theta_8)$.

Hence, for the four points of contact of a conic having quartic contact with the curve, we have

$$\phi(\alpha) \pm k_1\phi(\alpha') = 0, \quad \phi(\beta) \pm k_2\phi(\beta') = 0, \quad \phi(\gamma) \pm k_3\phi(\gamma') = 0,$$

which leads to three systems given by

$$\phi(\alpha) + k_1\phi(\alpha') = 0, \quad \phi(\beta) + k_2\phi(\beta') = 0, \quad \phi(\gamma) - k_3\phi(\gamma') = 0 \quad \ldots\ldots(4),$$

and two other sets of similar equations and a symmetrical system given by

$$\phi(\alpha) + k_1\phi(\alpha') = 0, \quad \phi(\beta) + k_2\phi(\beta') = 0, \quad \phi(\gamma) + k_3\phi(\gamma') = 0 \quad \ldots\ldots\ldots(5),$$

that is, there are four systems of conics having quartic contact with the curve. Again, for conics passing through a node and having triple contact with the curve, we put $\theta_7 = \alpha$, $\theta_8 = \alpha'$, say, in (3), and we get, for the three points of contact, two equations

$$\phi(\beta) \pm m_2\phi(\beta') = 0, \quad \phi(\gamma) \pm m_3\phi(\gamma') = 0 \ldots\ldots(6),$$

where m_2, m_3 are constants, which give four systems. Hence for the three nodes we have twelve such systems.

Again, for conics passing through two nodes and having double contact with the curve, we have for the two points of contact

$$\phi(\gamma) \pm n_3\phi(\gamma') = 0 \ldots\ldots\ldots\ldots\ldots(7),$$

that is, two systems, so that for the three pairs of nodes there are six systems of such conics.

Finally, there is the system of conics passing through the three nodes and touching the curve.

Let us consider one of the systems (4). Writing the curve in the form

$$ay^2z^2 + bz^2x^2 + cx^2y^2 + 2xyz(fx + gy + hz) = 0 \ldots(8),$$

where the triangle of reference is formed by the three nodes, this equation can be written $U^2 = z^2V$, where U is a conic

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passing through the nodes on z, so that one of the conics is of the form

$$z^2 + 2\theta U + \theta^2 V = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

Hence, since this conic is a perfect square for $\theta = 0$, the tangential coefficients are divisible by θ and are thus of the third degree in θ. Also the discriminant, after having been divided by θ^2, is of the form $f_1^2 \phi_2$. We thus obtain for the circular coordinates of a focus of (9)

$$\left.\begin{aligned} x &= \frac{G \pm f_1 \sqrt{\phi_2}\sqrt{b}}{G} \\ y &= \frac{F \pm f_1 \sqrt{\phi_2}\sqrt{a}}{G} \end{aligned}\right\} \ldots\ldots\ldots\ldots (10).$$

If a is proportional to b the curve satisfies two particular conditions, that is, two circles having double contact with the curve can be described through the nodes, and then the locus of the foci consists of two binodal quartics and two quintics, as we see by seeking the intersection of (10) with the line $\lambda x + \mu y + \nu = 0$, when we get

$$\begin{aligned} (\lambda G + \mu F - \nu C)^2 &= (\lambda + \mu^2)\, \Delta a \\ &= \lambda^2 (G^2 - \Delta C) + \mu^2 (F^2 - BC) + 2\lambda\mu \Delta a, \end{aligned}$$

whence

$$C f_2 + 2\lambda\mu\, (FG - \Delta a) = 0 \quad \ldots\ldots\ldots\ldots (11),$$

which, since $FG = CH + h\Delta$, is divisible by $h - a$. Now $C = a^2 - h^2$, and since it is of the third degree in θ one of the factors $a \pm h$ is of the second degree and the other factor $a \mp h$ is of the first degree. The two binodal quartics, which are circular, thus have their six foci in common with the given curve.

Now let us consider the circular trinodal quartic. Putting $b = \psi_1^2$, $a = \chi_1^2$ in (10), we have

$$x = \frac{G \pm f_1 \psi_1 \sqrt{\phi_2}}{C}, \quad y = \frac{F \pm f_1 \chi_1 \sqrt{\phi_2}}{C} \quad \ldots\ldots (12),$$

and these give two circular unicursal quartics and two unicursal quintics with I, J as double points. All these curves thus have their four foci in common with the given curve. Thus, considering the three systems of such generating conics, we have three pairs of confocal circular unicursal quartics, and this agrees with what we have proved already.

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Now let us consider the symmetrical system of conics having quartic contact with the trinodal quartic (8). This system can be written

$$V = Al^2x^2 + Bm^2y^2 + Cn^2z^2 + 2Fmnyz + 2Gnlzx + 2Hlmxy = 0,$$

where $A = bc - f^2$, $F = gh - af$, &c., that is, A, &c., are the tangential coefficients of the conic $(a, b, c, f, g, h)(x, y, z)^2$, and

$$l + m + n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

In this case, then, the tangential coefficients of V are of the fourth degree in a parameter, as

$$\Sigma = am^2n^2\lambda^2 + bn^2l^2\mu^2 + cl^2m^2\nu^2 + 2lmn(fl\mu\nu + gm\mu\lambda + hn\lambda\mu) = 0 \ldots (14).$$

For the discriminant Δ of V we have, then,

$$\sqrt{\Delta} = (abc + 2fgh - af^2 - bg^2 - ch^2)\, lmn = \Theta_2, \text{ say.}$$

Before proceeding to consider this system I shall show that the condition that the line joining two points on the curve should touch an inscribed conic breaks up into factors which can readily be found algebraically. We have seen already that this must take place for the conics that we have been considering, as C in (12) breaks up into factors for the circular curves, and C is the condition that the conic should touch the line at infinity, namely, the line joining the points I, J on the curve. Consider the curve

$$x : y : z = \alpha\beta : \gamma\delta : (\alpha\delta + \beta\gamma) \quad \ldots\ldots\ldots\ldots (15),$$

then $z^2 - 4xy\alpha(\alpha\delta - \beta\gamma)^2$, and so has quartic contact with the curve, if $\alpha\delta - \beta\gamma$ is of the fourth degree in a parameter. Hence, if the line joining two points on the curve touches the conic, we have

$$\{(\alpha\delta + \beta\gamma)(\alpha'\delta' + \beta'\gamma') - 2(\alpha\beta\gamma'\delta' + \alpha'\beta'\gamma\delta)\}^2 - (\alpha\delta - \beta\gamma)^2(\alpha'\delta' - \beta'\gamma')^2 = 0 \ldots (16),$$

which consists of the factors

$$(\alpha\gamma' - \alpha'\gamma)(\beta\delta' - \beta'\delta)(\alpha\delta' - \beta'\gamma)(\alpha'\delta - \beta\gamma').$$

For the three systems (9) $\alpha, \beta, \gamma, \delta$ are of the second degree in the parameter, and for the symmetrical system α, γ, say, are of the third degree, and β, δ are linear in the parameter.

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This investigation is apropos of the circular curve. For the coordinates of the foci of V, then, in circular coordinates, we have

$$x = \frac{G \pm \psi_1 \Theta_2}{C}, \quad y = \frac{F \pm \chi_1 \Theta_2}{C} \quad \ldots\ldots\ldots(17).$$

Hence, since $G^2 - \psi_1^2 \Theta_2^2 = AC$, $F^2 - \chi_1^2 \Theta_2^2 = BC$, and since C breaks up into factors, as we have seen, and therefore A, B also break up into factors, we have four curves given by equations of the form

$$\left.\begin{aligned} x &= \frac{f_1}{\phi_1}, & y &= \frac{f_1'}{\chi_1} \\ x &= \frac{f_2}{\chi_1 \psi_1 \omega_1}, & y &= \frac{f_2'}{\phi_1 \psi_1 \omega_1} \end{aligned}\right\} \ldots\ldots\ldots(18).$$

These evidently give a circle, a circular unicursal quartic, and two imaginary cubics with I, J respectively as double points. The circle has evidently no relation to the foci of the given curve, but the circular unicursal quartic is confocal with the given curve.

As a special case consider the circular lemniscate

$$\beta^2\gamma^2 \sin 2A + \gamma^2\alpha^2 \sin 2B + \alpha^2\beta^2 \sin 2C = 0 \ldots\ldots(19)$$

in trilinear coordinates, then the generating conic is

$$\frac{\lambda^2 \sin 2A}{l^2} + \frac{\mu^2 \sin 2B}{m^2} + \frac{\nu^2 \sin 2C}{n^2} = 0 \quad \ldots\ldots(20),$$

where $l + m + n = 0$.

Now the locus of the foci of this conic is

$$\sin A \sqrt{\left(\frac{\cos A}{\alpha S_1}\right)} + \sin B \sqrt{\left(\frac{\cos B}{\beta S_2}\right)} + \sin C \sqrt{\left(\frac{\cos C}{\gamma S_3}\right)} = 0 \quad \ldots\ldots(21),$$

where S_1, S_2, S_3 are the circles described on the sides as diameters respectively. This, it can readily be verified, is satisfied by points on the circumscribing circle. The remaining factors are then a confocal circular unicursal quartic and two imaginary cubics of the kind we have mentioned. Corresponding to the values $l = 0$, $m = 0$, $n = 0$ respectively of the parameter, the conic (13) breaks up into the lines

$$Cy^2 + Bz^2 - 2Fyz = 0, \ \&c,$$

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namely, the pairs of tangents drawn to the curve from the three nodes respectively. We see thus that the locus of each focus must pass through the three nodes. Hence the circle which we have found must in all cases be the circumscribing circle of the triangle formed by the nodes. Also a confocal circular unicursal quartic can be described to pass through the three nodes.

It may be observed that if we project this result, we see that one point of intersection of the tangents drawn from any two points of the curve I, J to the conic (13) lies on the conic passing through I, J and the three nodes. For the general trinodal quartic we have for the foci of the conic (13)

$$x = \frac{G + \Theta_3 \sqrt{b}}{C}, \quad y = \frac{F + \Theta_3 \sqrt{a}}{C} \quad \ldots\ldots\ldots(22).$$

If the curve be such that a is proportional to b, the locus consists of two unicursal sextics with I, J as double points. Further, if a and b are not restricted, we can find confocal curves by rationalising a and b by substitution, that is, by putting $\theta = \frac{\alpha - \beta u^2}{1 - u^2} = \frac{\gamma - \delta v^2}{1 - v^2}$, where α, β are the roots of a and γ, δ those of b. Assuming, then, a homographic relation connecting u and v, we get confocal unicursal curves, as, for instance, circular unicursal quintics, &c., I now proceed to consider one of the systems of conics passing through a node and having triple contact with the curve. Such a conic in relation to the quartic (8) will evidently have an equation of the form

$$V = \lambda^2 (x^2 + 2pzx) + \mu^2 (y^2 + 2qyz) + 2\lambda\mu (lyz + mzx + nxy) = 0 \ldots(23).$$

Hence the tangential coefficients are of the fourth degree in the parameter and the discriminant Δ is of the form

$$\lambda^2\mu^2 (\alpha\lambda^2 + \beta\mu^2 + \gamma\lambda\mu) \ldots\ldots\ldots\ldots(24),$$

in which it may be observed that $\gamma^2 = 4\alpha\beta$, if the node becomes a cusp.

We find, then, that the locus of the foci of V for the circular trinodal quartic consists of two unicursal sextics with I, J as double points, four single foci in common with the given curve, and a bitangent focus at the node xy. For the quartic with a cusp at xy these sextics seem to break up into

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conics and nodal bicircular quartics, having their foci at the cusp and the three foci of the given curve. If the curve is not circular, but is such that a is proportional to b, the locus of the foci of V consists of two sextics, each having 9 nodes, among which I, J are included. These sextics have a bitangent focus at the node and single foci at the six foci of the given curve. For, the systems of conics having double contact with the curve and passing through two nodes, the tangential coefficients are of the fourth degree and the discriminant is of the form $\theta^2\Theta_4$. Hence, for the locus of the foci of one of the systems in the case of the circular trinodal quartic, we get two sextics having 9 nodes, among which I, J are included. These sextics have bitangent foci at the two nodes and single foci at the four foci of the given curve.

The equation of a conic passing through the three nodes and touching the curve is evidently of the form

$$(a\theta^2 + b\theta + c)\, yz + (a'\theta^2 + b'\theta + c')\, zx + (a''\theta^2 + b''\theta + c'')\, xy = 0 \ldots (25),$$

or, if the quartic is circular, may be written, in trilinear coordinates,

$$\lambda\beta\gamma + \mu\gamma\alpha + \nu\alpha\beta = 0 \ldots\ldots\ldots\ldots\ldots (26),$$

where

$$(l\lambda + m\mu + n\nu) = \lambda^2 + \mu^2 + \nu^2 - 2\mu\nu\cos A - 2\nu\lambda\cos B - 2\lambda\mu\cos C.$$

We find, then, for the locus of the foci, in the case of the circular trinodal quartic, two sextics with 8 nodes, among which I, J are included. These sextics have bitangent foci at the three nodes, and single foci at the four foci of the given curve.

We might also consider the system of conics having double contact with the quartic and having double contact with a conic V which has quartic contact with the curve. If the quartic were such that it had quartic contact with a circle $V = xy - k^2$, then a conic having double contact with V and the curve could be written in the form in circular coordinates

$$ax + by + c + dV(xy - k^2) = 0 \ldots\ldots\ldots\ldots (27),$$

where a, b, c, d are quadratic expressions in a parameter θ. Since for the foci of this conic we have

$$x : y : z = b : a : (c + \theta\sqrt{\phi_2}) \ldots\ldots\ldots\ldots (28),$$

it follows that the locus of those collinear with xy is a unicursal quartic, which must thus be confocal with the given curve.

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I now proceed to consider curves confocal with a binodal quartic. Now for the binodal quartic $U^2 = z^2 V$, where U, V are conics and z is a line, there is the unique system of conics

$$S = \theta^2 V + 2\theta U + z^2 = 0 \quad \ldots\ldots\ldots\ldots (29),$$

having quartic contact with the curve. Since this is a perfect square for $\theta = 0$, the tangential coefficients are divisible by θ and so are of the third degree in θ; and the discriminant Δ is of the form $\theta^4 \Theta_4$. In fact the tangential equation of (29) can be written

$$\frac{\lambda^2}{\theta - a} + \frac{\mu^2}{\theta - b} + \frac{\nu^2}{\theta - c} + \frac{\rho^2}{\theta - d} = 0 \quad \ldots\ldots\ldots (30),$$

where $\theta - a$, &c., are the factors of Θ_4 and λ, μ, ν, ρ are the points of intersection of four pairs of double tangents. Now if the curve be such that it has one circle V belonging to the system (29) we have seen elsewhere that by considering the conics having double contact with V we can find a binodal quartic confocal with the curve. But if there are two circles belonging to the system, then for the locus of the foci of $S\,\{= (a, b, c, f, g, h)\,(x, y, 1)^2\}$, we get

$$x = \frac{G + \sqrt{(a\Theta_4)}}{C}, \quad y = \frac{F + \sqrt{'a\Theta_4)}}{C} \quad \ldots\ldots\ldots (31),$$

where the coordinates are circular and $a = b$.

This gives two circular uninodal quartics, which thus have their eight foci in common with the given curve. These two curves evidently cut orthogonally at the four points λ, μ, ν, ρ.

Again, for the locus of the foci of S in the case of the circular binodal quartic, we have

$$x = \frac{G + f_1\sqrt{\Theta_4}}{C}, \quad y = \frac{F + g_1\sqrt{\Theta_4}}{C} \quad \ldots\ldots\ldots (32),$$

which, by the process made use of at (11), are shown to give two circular binodal quartics having their six foci in common with the given curve, and cutting each other orthogonally at the four points λ, μ, ν, ρ. We thus have three general circular binodal quartics having their six foci in common.

I now consider other systems of conics having quartic contact with the curve. Let A, B, C, D be the four tangents drawn to the curve from a node, and A', B', C', D' be the corresponding tangents drawn from the other node, then if U

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is a conic passing through the two nodes the curve may be written $U^2 = ABA'B'$, which is the envelope of

$$V = \lambda^2 AB + 2\lambda\mu U + \mu^2 A'B' = 0 \ldots\ldots\ldots\ldots (33).$$

Hence, considering the pairs AB, $A'B'$, AB, $C'D'$, &c., as we have done for the case of the cubic, we get twelve systems of conics such as (33). Writing V in the form

$$\lambda^2 (x^2 - z^2) + \mu^2 (y^2 - z^2) + 2\lambda\mu (cz^2 + 2hxy + 2fyz + 2yzx) = 0 \ldots\ldots (34),$$

we see that Δ is of the form

$$\lambda^3\mu^3 (a\lambda^2 + \beta\mu^2 + \gamma\lambda\mu) \ldots\ldots\ldots\ldots (35).$$

Also the tangential coefficients are of the fourth degree in the parameter. We then find for the locus of the foci of V in the case of the circular curve

$$x = \frac{G + f_2 f_1 \sqrt{\phi_2}}{C}, \quad y = \frac{F + f_2 g_1 \sqrt{\phi_2}}{C} \ldots\ldots (36),$$

which give two unicursal sextics with I, J as double points, and having their six foci in common with the curve. Rationalising then ϕ_2 by substitution and making the suitable transformation, we can, as I have shown elsewhere, derive the equations of the systems of confocal and orthogonal unicursal sextics with I, J as double points. Hence, if the given curve satisfy a single particular condition, we can find two confocal and orthogonal unicursal quartics, these being particular members of the systems of confocal sextics.

We may also notice the case in which the binodal quartic has quartic contact with two circles belonging to the system (34). The locus of the foci of the system then consists of two sextics having 9 double points, among which I, J are included, and having their eight foci in common with the given curve.

I now consider conics passing through a node and having triple contact with the curve. If the curve referred to a node be written $u_4 + zu_3 + z^2 u_2 = 0$ and we eliminate z by putting $zv_1 + v_2 = 0$, for a conic through the node, we get an equation from which we can derive by Abel's theorem two relations of the form

$$\Sigma_6' L = \text{a constant}, \quad \Sigma_6' M = \text{a constant} \ldots\ldots (37),$$

connecting the six points of intersection, where L, M are the

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first class of hyperelliptic integrals. Hence if the conic has triple contact with the curve we have for the points of contact

$$\Sigma_3' L = \tfrac{1}{2}\Omega, \quad \Sigma_3' M = \tfrac{1}{2}\Omega' \quad \ldots\ldots\ldots\ldots(38),$$

so that by the theory of hyperelliptic integrals we have 16 such systems of conics. But in the case that we are discussing the curve has another node, and the hyperelliptic integrals are replaced by elliptic integrals of the first and third kind, which are equivalent to an elliptic integral of the first kind and an angle, so that we then have 8 distinct systems of such conics. For these conics the tangential coefficients are of the fourth degree in the parameter and the discriminant is of the form $f_1^{\,2}\Theta_4$. For the locus of the foci then, in the case of the circular binodal quartic, we find two sextics, having 9 double points, among which I, J are included, and having a bitangent focus at the node and single foci at the six foci of the given curve.

I now consider the system of conics passing through the two nodes and having double contact with the curve. These conics can be projected into circles having double contact with a bicircular quartic, there are thus four such systems of conics. The tangential coefficients are of the fourth degree in the parameter, and the discriminant is of the form Θ_6. Hence for the locus of the foci of the system, in the case of the circular binodal quartic, we find two sextics having 8 nodes, among which I, J are included, and having bitangent foci at the two nodes and single foci at the six foci of the given curve. These two sextics evidently cut orthogonally at four points of intersection of the tangents drawn to the curve from the nodes, respectively.

In the preceding cases we might give some consideration to the confocal curves when the binodal becomes a bicuspidal quartic. The conic (29) then becomes

$$\theta^2 P^2 + 2\theta U + PQ = 0 \quad \ldots\ldots\ldots\ldots(39)$$

so that it passes through the two cusps and has double contact with the curve. For the circular bicuspidal quartic the locus of the foci of (39) consists of two circular binodal quartics with their foci at the cusps and the four single foci of the curve.

For one of the other system of conics (33) Δ is of the form $\lambda^2\mu^2$ for the bicuspidal quartic, and the locus of the foci consists of confocal unicursal sextics with I, J as cusps. It may be observed that in the case of the Cartesian oval such

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conics as (33) are rigid and have their principal axes passing through fixed points.

VII. *On the determination of the nodo-bicuspidal quartic when the foci are given.*

A nodo-bicuspidal quartic is determined by 9 constants and being of the fourth class has four foci. Thus there is a singly infinite system of confocal curves, which I propose to determine. This curve possesses the property that its reciprocal is, in general, a curve of precisely the same kind; thus it has one bitangent and two inflexional tangents. Let x be a bitangent and y the line joining two cusps, then a quartic can be written $U^2 = xy^3$, where U is a conic. Thus, if z is the polar of xy with regard to U, the quartic is

$$(z^2 + ax^2 + by^2 + 2hxy)^2 - xy^3 = 0 \quad \ldots\ldots\ldots\ldots (1).$$

Now if this quartic has a double point it must lie on $z = 0$, and the discriminant of $(ax^2 + by^2 + 2hxy)^2 - xy^3 = 0$ vanishes, or, if we put $x = yt^2$, the discriminant of

$$at^4 + 2ht^2 - t + b = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

vanishes. This being divisible by a gives a relation of the fifth degree connecting a, b, h. Now let us use α, β, γ, the perpendiculars from the vertices of a triangle on a line, as tangential coordinates. We may write the tangential equation of the curve then

$$(\gamma^2 + A\alpha^2 + B\beta^2 + 2H\alpha\beta)^2 - M^3\alpha^2\beta = 0 \text{ or } \Sigma^2 - M^3\alpha^2\beta = 0$$

$$\ldots\ldots\ldots\ldots (3),$$

where β is the double point, α is the intersection of the inflexional tangents, γ is the intersection of the double tangent and the line joining the cusps, and the discriminant of

$$At^4 + 2Ht^2 - Mt + B = 0 \ldots\ldots\ldots\ldots\ldots (4)$$

vanishes, giving a relation of the fifth degree connecting A, B, H, M.

Now to find the foci of (3) we put $p - a_1 \cos\omega - b_1 \sin\omega$ for α, &c., and then take $\frac{p}{\cos\omega} = x$, y successively corresponding to $\tan\omega = \pm\sqrt{(-1)}$, where x, y are circular coordinates. We thus obtain

$$\{(x-\alpha'')^2 + A(x-\alpha)^2 + B(x-\alpha')^2 + 2H(x-\alpha)(x-\alpha')\}^2$$
$$- M^3(x-\alpha)^2(x-\alpha') = 0,$$
$$\{(y-\beta'')^2 + A(y-\beta)^2 + B(y-\beta')^2 + 2H(y-\beta)(y-\beta')\}^2$$
$$- M^3(y-\beta)^2(y-\beta') = 0,$$

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hich we may write

$$\left.\begin{aligned}(x^2-px+q)^2-m^2(x-\alpha)^2(x-\alpha')=0\\(y^2-p'y+q')^2-m^2(y-\beta)^2(y-\beta')=0\end{aligned}\right\} \quad \ldots\ldots\ldots(5).$$

Now let us make use of a particular case of Abel's theorem, amely, that if x_1, x_2, x_3, x_4 are the roots of

$$X-(\lambda x^2+\mu x+\nu)^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

here $$X=(x-a)(x-b)(x-c)(x-d),$$

ve have

$$u_1+u_2+u_3+u_4=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(7),$$

where $$u=\int\frac{dx}{\sqrt{X}},$$

We now consider the point α, β, the intersection of the inflexional tangents, as giving the parameter of the confocal system, for this is equivalent to a single parameter in consequence of α, β lying on a locus. Applying then (6) and (7) to (5) we get $u'=-3u$, and $v'=-3v$, where $u=\int\frac{dx}{\sqrt{X}}$ and $v=\int\frac{dy}{\sqrt{Y}}$, and X, Y give the circular coordinates respectively, of the four foci. This, according to the theory of elliptic functions, shows that the coordinates α', β' of the node are given rationally in terms of the coordinates of α, β by expressions of the form

$$\alpha'=\frac{P}{Q}, \quad \beta'=\frac{P'}{Q'} \quad \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

where P, Q; P', Q' are polynomials in α, β respectively, of the ninth degree. Again from (5) we see that the result of differentiating

$$x^2-px+q-\sqrt{(1-m^2)}\sqrt{X}=0 \ldots\ldots\ldots\ldots (9)$$

twice must be satisfied for $x=\alpha$, that is, α satisfies

$$2(2x-p)=\sqrt{(1-m^2)}\,\frac{X'}{\sqrt{X}} \quad \ldots\ldots\ldots(10),$$

$$8=\sqrt{(1-m^2)}\,\frac{\{2XX''-X'^2\}}{(X)^{\frac{3}{2}}} \ldots\ldots\ldots\ldots(11).$$

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Similarly β satisfies

$$y^2 - p'y + q' - \sqrt{(1-m^2)}\sqrt{Y} = 0,$$

$$2(2y-p) = \sqrt{(1-m^2)}\,\frac{Y'}{\sqrt{Y}} \quad \ldots\ldots\ldots\ldots (12),$$

$$8 = \sqrt{(1-m^2)}\,\frac{\{2YY'' - Y'^2\}}{Y^{\frac{1}{2}}}.$$

We thus get, for the locus of α, β,

$$\frac{(2XX'' - X'^2)^2}{X^3} - \frac{(2YY'' - Y'^2)^2}{Y^3} = 0 \quad \ldots\ldots (13),$$

a curve of the 22nd degree, as it is divisible by the square of the line at infinity. Also from (9), (10), (11), and (12) we determine p, q, p', q', m^2 in terms of α, β. Now p, q give the coordinates of the foci of the conic

$$\Sigma = \gamma^2 + A\alpha^2 + B\beta^2 + 2H\alpha\beta = 0 \ldots\ldots\ldots\ldots (14),$$

viz.

$$p = x_1 + x_2,\ q = x_1x_2, \text{ and } p' = y_1 + y_2,\ q' = y_1y_2;$$

that is, if Σ be written in circular tangential coordinates thus:

$$\Sigma = A\lambda^2 + B\mu^2 + \nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu \ldots (15),$$

the curve itself being

$$\Sigma^2 - m^2(\alpha\lambda + \beta\mu + \nu)^2(\alpha'\lambda + \beta'\mu + \nu) = 0,$$

we have

$$2G = p,\ 2F = p',\ A = q,\ B = q',$$

or

$$\left.\begin{aligned} G &= \frac{x(2XX'' - X'^2) - 2XX'}{2XX'' - X'^2} \\ A &= \frac{8X^3 - 4xXX' + x^2(2XX'' - X'^2)}{2XX'' - X'^2} \end{aligned}\right\} \ldots\ldots (16),$$

$$\left.\begin{aligned} F &= \frac{y(2YY'' - Y'^2) - 2YY'}{2YY'' - Y'^2} \\ B &= \frac{8Y^3 - 4yYY' + y^2(2YY'' - Y'^2)}{2YY'' - Y'^2} \end{aligned}\right\} \ldots\ldots (17),$$

Now $\gamma(\alpha'', \beta'')$, the intersection of the bitangent and the line joining the cusps, is the pole of the line joining the node

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and the intersection of the inflexional tangents with regard to Σ. Let the latter line be $lx+my+n=0$.

Then from (8) l, m, n can be expressed in terms of the coordinates of α, β, and α'', β'' are expressed thus

$$\alpha''=\frac{Al+Hm+Gn}{Gl+Fm+n},\quad \beta''=\frac{Hl+Bm+Fn}{Gl+Fm+n}\ldots(18).$$

But the condition that the curve should have a bitangent is expressed by taking the discriminant of the tangents drawn to the curve from α'', β'', namely,

$$\{(\alpha''\lambda+\beta''\mu)^2-2(\alpha''\lambda+\beta''\mu)(G\lambda+F\mu)+A\lambda^2+B\mu^2+2H\lambda\mu\}^2$$
$$=\{(\alpha-\alpha'')\lambda+(\beta-\beta'')\mu\}^2\{(\alpha'-\alpha'')\lambda+(\beta'-\beta'')\mu\}\ldots(19).$$

This gives an equation of the tenth degree for H, so that there are ten different confocal curves with their intersection of inflexional tangents at a given point of a curve of the 22nd degree (13).

The case in which the curve has an inflexional focus deserves special consideration. Taking the origin at that point we have $X=x^2(x-a)(x-b)$ for the foci. Hence, since

$$(x^2-px+q)^2-m^2x^2(x-\alpha)=(1-m^2)X\ldots(20),$$

where α, β are the coordinates of the node, we obtain $q=0$, and

$$(x-p)^2-m^2x(x-\alpha)=(1-m^2)(x-a)(x-b)\ldots(21).$$

Similarly, we get

$$(y-p')^2-m^2y(y-\beta)=(1-m^2)(y-a')(y-b')\ldots(22).$$

Eliminating p, p', we have

$$(a+b+t\alpha)^2=4ab(1+t),\quad (a'+b'+t\beta)^2=4a'b'(1+t)\ldots(23),$$

where $t=\frac{m^2}{1-m^2}$. Hence, putting $1+t=\theta^2$, we get

$$\alpha:\beta:1=\{\pm 2\theta\sqrt{(ab)}-a-b\}:\{2\theta\sqrt{(a'b')}-a'-b'\}:(\theta^2-1)$$
$$\ldots\ldots\ldots(24),$$

so that the node lies on one or other of two conics.

Now since $q=0$, one focus of the conic Σ coincides with the inflexional focus, and the other focus is at the point p, p', for which we have, from (21), (22),

$$p^2=m^2ab,\quad p'^2=m^2a'b'\ldots\ldots\ldots\ldots\ldots(25);$$

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that is, the other focus moves on one or other of two fixed mutually rectangular lines passing through the given focus. Let N be the node and F the fixed focus, and $G(\alpha', \beta')$ the pole of FN with regard to Σ, then we find $\alpha'\beta + \beta'\alpha = 0$, and if $\Sigma = \varpi(\varpi - p\cos\omega - p'\sin\omega) - k^2$), we determine k^2 by expressing that α', β'; α, β are conjugate with regard to Σ, or that CN and FN are conjugate. We then express that the discriminant of

$$\{(\alpha' + \beta' t)\{(\alpha' - p) + (\beta' - p')t\} - k^2(1 + t^2)\}^2 - m^2(\alpha' + \beta' t)^2\{\alpha' - \alpha + (\beta' - \beta)t\} = 0 \ldots\ldots(26)$$

with regard to t vanishes.

It may be observed that the proceeding process is applicable to the cuspidal cubic. Suppose the cubic is written in tangential coordinates $\gamma^3 - m\alpha^2\beta = 0$, then for the foci we have

$$(x - \alpha'')^2 - m(x - \alpha)^2(x - \alpha') = (1 - m)X \ldots(27),$$

where $X = (x - a)(x - b)(x - c)$ gives the three foci, and α, β are the coordinates of the point of inflexion, and α', β' are those of the cusp. Now, by a particular case of Abel's Theorem, if $u = \int \frac{dx}{X^{\frac{2}{3}}}$, we have, from (27), $2u + u' = \omega$, a constant, where u corresponds to α, and u' to α'. We have also $2v + v' = \omega'$, where $v = \int \frac{dy}{Y^{\frac{2}{3}}}$.

Hence it follows that α', β' can be expressed in terms of α, β by relations of the form

$$\alpha = \frac{P}{Q}, \quad \beta' = \frac{P'}{Q'} \ldots\ldots\ldots\ldots(28),$$

where P, Q are of the fourth degree in α, and P', Q' are of the fourth degree in β, and this can be done in three ways; in fact P, Q involve a cube root of unity, so that we obtain three distiuct values of α of the form (28), and similarly for β. Again α must satisfy the result of differentiating

$$x - \alpha'' - (1 - m)^{\frac{1}{3}}X^{\frac{1}{3}} = 0 \ldots\ldots\ldots\ldots(29)$$

with regard to x. This gives

$$3(1 - m)^{-\frac{1}{3}} = X'X^{-\frac{2}{3}} \ldots\ldots\ldots\ldots(30),$$

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and similarly we have

$$3(1-m)^{-\frac{1}{3}} = Y'Y^{-\frac{2}{3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (31),$$

whence eliminating m we obtain

$$\frac{X'^3}{X^2} = \frac{Y'^3}{Y^2},$$

which by taking

$$X = x^3 + 3ax + b, \quad Y = y^3 + 3a'y + b'$$

gives

$$\frac{(x^2+a)^3}{(x^3+3ax+b)^2} = \frac{(y^2+a')^3}{(y^3+3a'y+b')^2} \quad \ldots\ldots\ldots (32);$$

which is divisible by the square of the line at infinity, that is, the locus of the point of inflexion is a curve of the tenth degree.

Again from (30) and (31) we have

$$\left.\begin{aligned} \alpha'' &= -\frac{2ax+b}{x^2+a}, \\ \beta'' &= -\frac{2a'y+b'}{y^2+a'} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (33).$$

Also

$$-m = \frac{3ax^4 + 6a^2x^2 + 2b(x^3+3ax) + b^2 - a^3}{(x^2+a)^3} \ldots (34).$$

These equations completely determine three cuspidal cubics with their foci at given points and having a common point of inflexion at a given point P on a certain locus of the tenth degree (32).

I now return to the nodo-bicuspidal curve and I observe that if the node and the intersection of the inflexional tangents are at infinity the foci form a parallelogram, and that there is still a singly infinite system of confocal curves. The tangential equation of the curve in p, ω coordinates may then evidently be written

$$(p^2 - a^2\cos^2\omega - b^2\sin^2\omega)^2 - m^4\cos^3(\omega-\alpha)\cos(\omega-\beta) = 0$$

$\ldots\ldots\ldots (35),$

where the origin is at the point of intersection of the bitangent and the line joining the cusps, and the discriminant of

$$(a^2\cos^2\omega + b^2\sin^2\omega)^2 - m^4\cos^3(\omega-\alpha)\cos(\omega-\beta) = 0 \ldots (36)$$

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vanishes. The foci of (35) are evidently found from

$$x^2 = a^2 - b^2 \pm m^2 e^{i\theta}, \quad y^2 - a^2 - b^2 \pm m^2 e^{-i\theta} \quad \ldots\ldots(37),$$

where x, y are circular coordinates, and

$$3\alpha + \beta = 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots(38).$$

Thus they form a parallelogram with its intersection of diagonals at the origin and are completely determined when $a^2 - b^2$, m^2, and $3\alpha + \beta$ are given. Hence if we put

$$a^2 - h^2, \ b^2 - h^2 \text{ for } a^2, b^2,$$

respectively, where a, b are given, and make the discriminant of

$$\{(a^2 - h^2)\cos^2\omega + (b^2 - h^2)\sin^2\omega\}^2 - m^4\cos^2(\omega - \alpha)\cos(\omega - \beta) = 0 \quad \ldots\ldots\ldots(39)$$

vanish, we get a relation connecting h^2, α, β, which with $3\alpha + \beta$ given determines the singly infinite system of confocal curves.

I shall now show that for every curve of the confocal system there is another curve of the system cutting the given curve orthogonally. I have shown elsewhere and it is susceptible of an easy proof that the curves

$$\left.\begin{aligned} p^2 &= C + f(\omega) \\ p^2 &= C - f(\omega + \tfrac{1}{2}\pi) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(40)$$

cut orthogonally. Hence the curves

$$\left.\begin{aligned} \{p^2 - (a^2 - h^2)\cos^2\omega - (b^2 - h^2)\sin^2\omega\}^2 \\ = m^4\cos^2(\omega - \alpha)\cos(\omega - \beta) \\ \{p^2 + (a^2 - h'^2)\sin^2\omega + (b^2 - h'^2)\cos^2\omega\}^2 \\ = m^4\sin^2(\omega - \alpha)\sin(\omega - \beta) \end{aligned}\right\} \quad \ldots\ldots(41)$$

cut orthogonally. These curves are confocal since

$$3\alpha' + \beta' = 3\alpha + \beta.$$

Again, the discriminants of

$$\left.\begin{aligned} \{(a^2 - h^2)\cos^2\omega + (b^2 - h^2)\sin^2\omega\}^2 \\ - m^4\cos^2(\omega - \alpha)\cos(\omega - \beta) = 0 \\ \{(a^2 - h'^2)\sin^2\omega + (b^2 - h'^2)\sin^2\omega\}^2 \\ - m^4\sin^2(\omega - \alpha)\sin(\omega - \beta) = 0 \end{aligned}\right\} \quad \ldots\ldots(42)$$

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vanish. But the second of these equations will coincide with the result of substituting $\omega + \frac{1}{2}\pi$ for ω in the first, if

$$a^2 + b^2 = h^2 + k'^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(43),$$

so that subject to this condition the discriminants, being made to vanish, express the same relation. Thus the two curves of the confocal system connected by the relation (43) cut each other orthogonally.

It may be observed that in the preceding investigation we have arrived at the solution of the more general problem: to find the system of curves of the fourth class with two inflexional tangents, when the four foci are given.

Such a curve involves ten constants, so that the confocal system involves two parameters. The investigation is then precisely that which we have followed up to (17), the discriminant of (19) not being made to vanish. The parameters are thus the constant H in (15) and the point α, β on the curve (13).

We can thus find more special singly infinite confocal systems corresponding to some singularity of the curve which connects H with α, β. In the case that the foci form a parallelogram and the node and intersection of inflexional tangents are at infinity, the doubly infinite confocal system is given by

$$\{p^2 - (a^2 - h^2)\cos^2\omega - (b^2 - h^2)\sin^2\omega\}$$
$$= m^2\cos^2(\omega - \alpha)\cos(\omega + 3\alpha - 2\theta)\ldots\ldots(44),$$

where h^2 and α are the parameters.

We see then from (40) that two confocal curves cut orthogonally if $\alpha - \alpha' = \frac{1}{2}\pi$.

VIII. *On a confocal system of nodal cubics.*

If a plane nodal cubic touch the line at infinity the curve is determined by seven constants and has three foci. Consequently, when the three foci are given, the curve belongs to a singly infinite system. I have shown elsewhere that this system can be expressed in circular coordinates thus:

$$\left.\begin{aligned} 3\theta x &= a\theta^3 + 3b\theta^2 + 3c\theta + d \\ 3\phi y &= a'\phi^3 + 3b'\phi^2 + 3c'\phi + d' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1),$$

where $\phi = \lambda\theta$, λ being the parameter of the system, and θ being the parameter of a point on a given curve of the system. It is, in fact, evident that the foci of (1) are determined by equations independent of λ, namely, the discriminants of the

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binary cubic equations (1) in θ and ϕ, respectively. In these equations (1) it is evident that $\theta=\infty$ corresponds to the parabolic point at infinity and $\theta=0$ to the point of contact of the asymptote.

Altering the origin in (1) we can evidently make the coefficients of θ and ϕ vanish; and then multiplying θ and ϕ by certain factors we can express the curve thus:

$$\left.\begin{aligned} 3\theta x &= \theta^3 + 3m\theta^2 + 1 \\ 3\phi y &= \phi^3 + 3n\phi^2 + 1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where $\phi=\lambda\theta$.

If in this equation $m=n$, it is evident the values of x and y for the foci are determined by the same equation, so that the foci are then collinear, lying on the line $x=y$.

If two of the foci so determined are conjugate imaginary points, we must take their antipoints, which are then real. In this case the real foci form an isosceles triangle. Further, if

$$m^2=10\pm6\sqrt{3}, \text{ whence } m=1\pm\sqrt{3} \ldots\ldots\ldots(4),$$

one focus is the middle point of the other two. Again, if the foci form an equilateral triangle, taking the origin at the centre, we have

$$\left.\begin{aligned} -3\theta x &= \theta^3 - 6\theta^2 + 3\theta + 1 \\ -3\phi y &= \phi^3 - 6\phi^2 + 3\phi + 1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(5),$$

where $\phi=\lambda\theta$, the foci being given by $4x^3=4y^3=27$.

In the same case and with the same foci there is the system of confocal cuspidal cubics

$$\theta x=1+\theta^3, \quad \phi y=1+\phi^3 \quad \ldots\ldots\ldots\ldots\ldots(6),$$

where $\phi=\lambda\theta$, with the cusp and inflexion at infinity.

From (2) we have

$$3(x-\lambda y)=\theta^2(1-\lambda^3)+3\theta(m-n\lambda^2) \quad \ldots\ldots(7),$$

from which it follows that

$$x-\lambda y=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

is the asymptote of the curve. We see thus that the asymptotes of the confocal system all pass through a fixed point, namely, the origin determined in the equations (2). Again the tangent of the curve drawn parallel to the asymptote is, from (7),

$$x-\lambda y=\tfrac{3}{4}\frac{(m-n\lambda^2)^2}{\lambda^3-1} \quad \ldots\ldots\ldots\ldots\ldots(9),$$

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and this line touches a unicursal curve of the fourth class with the line at infinity as a triple tangent and the origin as a double point. When $m=n$ this envelope reduces to a tricuspidal quartic with the line at infinity as a bitangent. Again from (2) we get

$$3\theta\lambda(\lambda^2 x - y) = 3\theta^2\lambda^2(m\lambda - n) + \lambda^3 - 1 \quad \ldots\ldots(10),$$

from which we find that the equations of the two tangents parallel to the axis of the parabola touching the curve at infinity are

$$\lambda^2 x - y = \pm \frac{2}{\sqrt{3}} \sqrt{\{(\lambda^3 - 1)(m\lambda - n)\}} \quad \ldots\ldots(11),$$

and these lines touch the curve of the fourth class

$$(3\gamma^2 - 4m\alpha^3 - 4n\beta^3)^2 - 16\alpha\beta(n\alpha - m\beta)^3 = 0 \ldots(12)$$

with two double tangents. If $m=n$, this envelope is unicursal.

From (2) we find that the condition that three points on the curve given by

$$\theta^3 - p_1\theta^2 + p_2\theta - p_3 = 0 \quad \ldots\ldots\ldots\ldots(13)$$

should be collinear is

$$p_1(\lambda^3 - 1) + 3n\lambda^2(1 + p_2) - 3m(1 + \lambda^3 p_3) = 0 \ldots(14).$$

Hence for the three points of inflexion we have

$$n\lambda^2(1 + \theta^3) - m(1 + \lambda^3\theta^3) + (\lambda^3 - 1)\theta = 0 \ldots(15),$$

which lie on the line

$$3\lambda(n\lambda x - my) + \lambda^3 - 1 = 0 \quad \ldots\ldots\ldots(16).$$

We see thus that the line of inflexions of the confocal system touches a tricuspidal hypocycloid having the intersection of the cuspidal tangents at the origin; for, putting $\lambda = e^{2i\omega}$ (16) becomes in rectangular coordinates

$$x\cos(\omega - \alpha) + y\sin(\omega - \alpha) = k\sin 3\omega.$$

With regard to the directions of the asymptote (8), of the parabolic point at infinity (11), and of the line of the inflexions (16), I observe that we have

$$e^{2i\omega_1} = \lambda, \quad e^{2i\omega_2} = \frac{1}{\lambda^2}, \quad e^{2i\omega_3} = \frac{m}{n\lambda},$$

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respectively. Thus the directions are connected by the relations

$$2\omega_1 = -\omega_2, \quad \omega_1 + \omega_2 = \alpha \quad \ldots\ldots\ldots\ldots\ldots(17),$$

where α is a constant.

With regard to the parameters of the node, we have from (2)

$$\left.\begin{aligned} \theta_1 + \theta_2 + 3m &= \frac{1}{\theta_1\theta_2} \\ \lambda(\theta_1 + \theta_2) + 3n &= \frac{1}{\lambda^2\theta_1\theta_2} \end{aligned}\right\} \ldots\ldots\ldots\ldots(18),$$

so that θ_1, θ_2 are given by

$$\theta^2 - \frac{3\theta(m - n\lambda^2)}{\lambda^3 - 1} + \frac{\lambda^3 - 1}{3\lambda^2(m\lambda - n)} = 0 \ldots\ldots\ldots(19),$$

and the curves of the system with cusps are determined by

$$27\lambda^2(m\lambda - n)(m - n\lambda^2)^2 - 4(\lambda^3 - 1)^3 = 0 \ldots\ldots(20);$$

that is, there are nine curves of the system with cusps, namely, nine curves, each of which passes through one of the nine foci, real and imaginary, so that there are three real curves.

To find the locus of the nodes of the system, we have

$$\left.\begin{aligned} \theta^3 + 3m\theta^2 - 3x\theta + 1 &= 0 \\ \lambda^3\theta^3 + 3n\lambda^2\theta^2 - 3\lambda y\theta + 1 &= 0 \end{aligned}\right\} \ldots\ldots\ldots(21);$$

whence

$$\left.\begin{aligned} (\lambda^3 - 1)\theta^2 + 3(n\lambda^2 - m)\theta + 3(x - \lambda y) &= 0 \\ 3\lambda^2(m\lambda - n)\theta^2 - 3\lambda(\lambda^2 x - y)\theta + \lambda^3 - 1 &= 0 \end{aligned}\right\} \ldots(22),$$

and these equations in θ must coincide for the node. We thus get

$$x - \lambda y = \frac{(\lambda^3 - 1)^2}{9\lambda^2(m - n\lambda)},$$

$$\lambda^2 x - y = \frac{3\lambda(n\lambda^2 - m)(n - m\lambda)}{\lambda^3 - 1};$$

whence

$$\left.\begin{aligned} x(\lambda^3 - 1) &= \frac{3\lambda^2(n\lambda^2 - m)(n - m\lambda)}{\lambda^3 - 1} + \frac{(\lambda^3 - 1)^2}{9\lambda^2(n - m\lambda)} \\ y(\lambda^3 - 1) &= \frac{3\lambda(n\lambda^2 - m)(n - m\lambda)}{\lambda^3 - 1} + \frac{(\lambda^3 - 1)^2}{9(n - m\lambda)} \end{aligned}\right\} \ldots(23);$$

that is, the locus is a unicursal curve of the ninth degree.

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If $m=n$ this locus reduces to the sixth degree. To find the envelope of the inflexional tangents we have from (2)

$$3\theta\lambda(\alpha x+\beta y+\gamma)=\lambda\alpha(\theta^3+3m\theta^2+1)$$
$$+\beta(\lambda^3\theta^3+3n\lambda^2\theta^2+1)+3\gamma\lambda\theta),$$

so that making the expression on the right a perfect cube we get

$$\gamma(\alpha+\lambda^2\beta)=(m\alpha+n\lambda\beta)^2 \quad \ldots\ldots\ldots\ldots(24),$$
$$\lambda\gamma^2=(m\alpha+n\lambda\beta)(\beta+\lambda\alpha) \quad \ldots\ldots\ldots\ldots(25),$$

Eliminating λ from these equations, we get the envelope

$$\alpha\beta\gamma\{mn\gamma-m\beta-n\alpha\}^2=(m^2\alpha+n^2\beta-\gamma)$$
$$\times\{(m\alpha^2+n\beta^2-\gamma^2)^2-4mn\alpha^2\beta^2\}\ldots\ldots(26),$$

which represents a curve of the fifth class passing through the five points given by the expression on the right.

We have also in addition to (24) and (25),

$$\lambda\gamma(m\alpha+n\lambda\beta)=(\alpha+\lambda^2\beta)(\beta+\lambda\alpha) \quad \ldots\ldots(27).$$

Hence from the parabolæ (24) and (27) which have fixed foci at the points

$$x=-m^2, \quad y=-n^2,$$

and
$$l+mx=0, \quad l+ny=0$$

respectively, we get for the locus of the foci of parabolæ touching their common tangents, the circle

$$(2Gx-A)(2F'y-B')=(2G'x-A')(2Fy-B),$$

namely,

$$\lambda(1+mx)(y+n^2)-(x+m^2)(l+ny)=0\ldots(28);$$

that is, the circle circumscribing the triangle formed by the inflexional tangents passes through two fixed points, namely, the foci of the parabolae (24) and (27).

Again, the directrices of the parabolæ (24) and (27) are

$$y+\lambda^2x+2mn\lambda=0,$$

$$m\lambda y+n\lambda^2x+1+\lambda^4=0 \quad \ldots\ldots\ldots\ldots(29)$$

respectively, which touch a circle and a tricuspidal hypo-

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cycloid. Now the intersection of the directrices of these parabolae is the intersection of the perpendiculars of the triangle formed by their common tangents; that is, the locus of the intersection of the perpendiculars of the triangle formed by the inflexional tangents is the unicursal curve of the fifth order

$$\left.\begin{aligned} x &= -\frac{1+\lambda^3-m^2n\lambda^2}{\lambda^2(n-m\lambda)} \\ y &= \frac{1+\lambda^3-mn^2\lambda}{n-m\lambda} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(30).$$

From these equations and the coordinates of the centre of the circle (27), we can obtain the locus of the centre of gravity, and that of the centre of the nine-points circle of the triangle formed by the inflexional tangents.

From (25), which may be written

$$\gamma^2 = m\alpha^2 + n\beta^2 + \frac{(n\lambda^2+m)}{\lambda}\alpha\beta \quad \ldots\ldots\ldots\ldots(31),$$

we see that the triangle formed by the inflexional tangents is circumscribed about a conic with given foci, namely, the points $x^2=m$, $y^2=n$.

Now the polar circle of the triangle cuts orthogonally the director circle of (31), namely,

$$xy = \frac{2(m+n\lambda^2)}{\lambda};$$

so that from (30) the polar circle is

$$xy\lambda^2(n-m\lambda) - \lambda^2(1+\lambda^3-mn^2\lambda)\,x + (l+\lambda^3-m^2n\lambda^2)\,y + 2\lambda(n-m\lambda)(m+n\lambda^2) = 0 \ldots\ldots(32).$$

If we seek the locus of the intersection of the tangents drawn to the system of parabolæ touching the common tangents of (24) and (27), parallel to the lines $x^2-y^2=0$, we get

$$(x+\lambda^2y+m+n\lambda^2)(m\lambda y+n\lambda^2x+1+\lambda^3) - (\lambda^2x+y+2mn\lambda)(m\lambda x+n\lambda^2y+\lambda+\lambda^3) = 0 \ldots\ldots(33).$$

Now this conic passes through the vertices of the triangle formed by the inflexional tangents. Hence we find the locus of the vertices of that triangle by substituting the value of λ obtained from (27) in (33).

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If $m=n=1$ in (2) two foci coincide and form an inflexional focus, this focus being $x=y=-1$, and the single focus being $x=y+\frac{5}{4}$.

The equation (15) giving the points of inflexion in this case is

$$\lambda^2(1+\theta^3)-(1+\lambda^3\theta^3)+(\lambda^3-1)\theta=0,$$

or
$$(\theta+1)(\lambda\theta+1)(\lambda+1-\lambda\theta)=0.$$

Now the first two factors give the imaginary inflexions, and the factor $\lambda+1-\lambda\theta=0$ the real point of inflexion. Hence substituting this value of θ in

$$x+1=\frac{(\theta+1)^3}{3\theta},\quad y+1=\frac{(\lambda\theta+1)^3}{3\lambda\theta}\quad\ldots\ldots(34),$$

we get, for the locus of the real point of inflexion

$$x+1=\frac{(2\lambda+1)^3}{3\lambda^2(\lambda+1)},\quad y+1=\frac{(\lambda+2)^3}{3(\lambda+1)}\quad\ldots\ldots(35),$$

a unicursal curve of the fifth order having the line at infinity as a bitangent, and an inflexional focus in common with the given system.

Again, for the envelope of the real inflexional tangent, we have, in this case, from (26), putting $m=n=1$ and dividing out by $(\alpha+\beta-\gamma)^2$,

$$\alpha\beta\gamma+(\gamma+\alpha+\beta)(\gamma+\beta-\alpha)(\gamma+\alpha-\beta)=0\ldots(36);$$

or, as we obtain from (24) and (26) by putting $m=n=1$,

$$\alpha=\lambda^3(\lambda+2),\ \beta=-(2\lambda+1),\ \gamma=(1-\lambda)(1+\lambda)^2\ldots(37);$$

that is, the envelope of the inflexional tangent is a tricuspidal quartic passing through the circular points.

It may be observed that if we take the origin at the inflexional focus, the curve (34) can be transformed by the substitution $x=X^3$, $y=Y^3$ into a cuspidal cubic of the confocal system defined by the equations (6).

If a nodal cubic have the line at infinity as an inflexional tangent, it has two finite foci and is determined by six constants. Consequently, if the foci are given, the curve contains two indeterminate parameters.

Such a system can be represented by

$$x=\theta^3-3\theta,\quad y=\phi^3-3\phi\quad\ldots\ldots\ldots\ldots(38)$$

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in circular coordinates, where $\phi = \lambda\theta + \mu$, λ, μ, being the parameters, and $x = y = \pm 2$ the given foci.

Now if λ be given the direction of the point at infinity is given, and since $e^{2iv} = \frac{\lambda}{\lambda'}$, where v is the angle of intersection of the curves, $\phi = \lambda\theta + \mu$, $\phi = \lambda'\theta + \mu$; if we consider λ as a constant and μ as a variable, we have a confocal system cutting at constant angles entire systems of confocal curves, and in particular, it may be remarked, cutting orthogonally the confocal system $\phi = -\lambda\theta + \mu'$

To find the locus of the nodes of this system we have, if

$$\theta_1 + \theta_2 = p, \quad \theta_1\theta_2 = q, \quad \phi_1 + \phi_2 = p', \quad \phi_1\phi_2 = q',$$

$$p^2 - q = 3, \quad p'^2 - q' = 3,$$

$$2x = p\,(p^2 - 3q - 3), \quad 2y = p'\,(p'^2 - 3q' - 3):$$

also

$$(\phi_1 - \phi_2)^2 = \lambda^2\,(\theta_1 - \theta_2)^2,$$

whence

$$p'^2 - 4q' = \lambda^2\,(p^2 - 4q).$$

We thus get

$$x = p\,(3 - p^2), \quad y = p'\,(3 - p'^2),$$

so that, putting $p^2 - 4 = u$, $p'^2 - 4 = v$, we have

$$x^2 = -(u+1)\,(u+4), \quad y^2 = -(v+1)\,(v+4) \ldots(39),$$

where $v = \lambda^2 u$, that is, the locus of the nodes is a curve of the fourth order, with two nodes at infinity, which, if λ varies, belongs to a confocal system.

Again, to find the finite inflexions we have for the points where the line $\alpha x + \beta y + \gamma = 0$ meets the curve

$$(\alpha + \beta\lambda^3)\theta^3 + 3\lambda^2\mu\beta\theta^2 + 3\,\{\beta\lambda\,(\mu^2 - 1) - \alpha\}\,\theta + \beta\,(\mu^3 - 3\mu) + \gamma = 0 \ldots\ldots\ldots(40),$$

so that if this equation is

$$\theta^3 - p_1\theta^2 + p_2\theta - p_3 = 0,$$

we get

$$p_1\,(\lambda^2 + \mu^2 - 1) + p_2\lambda\mu + 3\lambda\mu = 0 \ldots\ldots\ldots(41),$$

whence for the two inflexions we get

$$\lambda\mu\theta^2 + (\lambda^2 + \mu^2 - 1)\,\theta + \lambda\mu = 0 \ldots\ldots\ldots(42).$$

But substituting $\mu = \phi - \lambda\theta$ in this we get

$$\theta\,(\phi^2 - 1) - \lambda\phi\,(\theta^2 - 1) = 0 \ldots\ldots\ldots\ldots(43),$$

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which with

$$x = \theta^3 - 3\theta, \quad y = \phi^3 - 3\phi$$

gives a curve of the twelfth degree, of deficiency unity, confocal with the given curve.

Further for the inflexional tangents making (40) a perfect cube we get

$$\lambda\mu^2\alpha\beta = (\alpha + \lambda\beta)(\alpha + \lambda^2\beta),$$

$$(\beta\lambda\mu^2 - \alpha - \lambda\beta)^3 = \lambda^3\beta^3(\mu^4 - 3\mu^2) + \lambda^2\mu\beta\gamma \quad \ldots(44),$$

from which by the elimination of μ we obtain

$$(\alpha + \lambda\beta)\{\alpha^2 + 2\lambda(\lambda^2 - 1)\alpha\beta - \lambda^4\beta^2\}^2 = \lambda^3\alpha\beta(\alpha + \lambda^2\beta)\gamma^3 \ldots(45)$$

a curve of the fifth class with the line at infinity as a triple tangent.

If we consider the special curves corresponding to $\lambda = \pm 1$ we find that they have their direction at infinity parallel or perpendicular to the line joining the foci, and from (39) that the nodes of the system lie on the line joining the foci and on the line bisecting at right angles these points, respectively. Again, for the inflexions, we have from (43)

$$1 \pm \theta\phi = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(46),$$

so that the locus of these points is the curve obtained from (38) combined with (46), namely, special curves of the system of sextic curves cutting the system $\phi = \lambda\theta$ orthogonally.

Further the envelope of the inflexional tangents is, from (45),

$$(\alpha^2 - \beta^2)^3 \pm \alpha\beta\gamma^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(47),$$

the real curve belonging to which is a quadricuspidal hypocycloid, namely, the envelope of a line of given length moving with its ends on two rectangular lines.

With respect to the curves cutting orthogonally the general confocal system (2) I observe that they are obtained from

$$3\theta x = \theta^3 + 3m\theta^2 + 1,$$

$$3\phi y = \phi^3 + 3n\phi^2 + 1,$$

where

$$\phi\theta = \mu \quad \ldots\ldots\ldots\ldots\ldots\ldots(48).$$

These equations represent unicursal quartics with the circular points as the points of contact of a bitangent, and have their foci in common with the given system. If we put

$$\frac{\phi}{\theta} = \lambda = e^{iu}, \quad \phi\theta = e^{iv},$$

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u, v both satisfy the equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0 \quad \text{.....(49)}$$

in rectangular coordinates, and are conjugate functions.

The same will be the case for all the conjugate orthogonal systems of confocal curves that I have investigated elsewhere. Hence all these systems form solutions of physical problems depending on the differential equation (49), namely, they constitute solutions of problems arising in the theory of the irrotational motion of an incompressible fluid, the strain of an elastic body, etc.

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